Toric non-abelian Hodge theory II

joint with Nick Proudfoot

Tamás Hausel

IST Austria & EPF Lausanne
http://hausel.ist.ac.at

Hitchin 70: Celebrating 30 years of Higgs bundles and
15 years of generalized geometry
ICMAT, Madrid
Sept. 2016

European Research Council
Established by the European Commission
SwissMAP
The Mathematics of Physics
National Centre of Competence in Research
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
IST AUSTRIA
Institute of Science and Technology

Non-abelian Hodge theory

- Simpson (1990), Hitchin (1987) for Riemann surfaces
- G complex reductive algebraic group, e.g. $\mathrm{G} = \mathrm{GL}_n(\mathbb{C})$
  C smooth complex projective curve (w. decorations)
- $\mathcal{M}_{\mathrm{B}} := H^1_{\mathrm{B}}(C, \mathrm{G}) = \left\{ \begin{array}{c} \text{moduli of representations} \\ \text{of } \pi_1(C) \to \mathrm{G} \end{array} \right\}$
- $\mathcal{M}_{\mathrm{DR}} := H^1_{\mathrm{DR}}(C, \mathrm{G}) = \{\text{moduli of flat } \mathrm{G}\text{-connections on } C\}$
- $\mathcal{M}_{\mathrm{Dol}} := H^1_{\mathrm{Dol}}(C, \mathrm{G}) = \{\text{moduli of } \mathrm{G}\text{-Higgs bundles on } C\}$
- Non-Abelian Hodge Theorem: $\mathcal{M}_{\mathrm{Dol}} \cong_{diff} \mathcal{M}_{\mathrm{DR}} \overset{\tau_{RH}}{\cong}_{an} \mathcal{M}_{\mathrm{B}}$
- Hitchin map: $\begin{array}{ccc} \chi : \mathcal{M}_{\mathrm{Dol}} & \to & \mathcal{A} \\ (E, \phi) & \mapsto & CharPol(\phi) \end{array}$
  proper, integrable system
- often $0 \in \mathcal{A}$ when $\chi^{-1}(0) \sim \mathcal{M}_{\mathrm{Dol}}$ nilpotent cone
- $C \cong \mathbb{P}^1 \rightsquigarrow \mathcal{M}^*_{\mathrm{DR}} := \{\text{moduli of flat connections on } C \times \mathrm{G}\}$
- $\mathcal{M}^*_{\mathrm{DR}} \subset \mathcal{M}_{\mathrm{DR}}$ open, $\mathcal{M}^*_{\mathrm{DR}} \cong Q$ star-shaped quiver variety

Conjectures on MHS on $H^*(\mathcal{M}_{\mathrm{B}})$

Conjecture

1. *(Hodge-Tate)*
   $h^{p,q}(H^*(\mathcal{M}_B)) \neq 0 \Rightarrow p = q$
2. *(Curious Hard Lefschetz)*
   $\alpha := [\Re(\Omega)] \in H^{2;2,2}(\mathcal{M}_{\mathrm{B}})$

$$L^l : \quad \begin{array}{ccc} Gr^W_{\dim -2l} H^{i-l}(\mathcal{M}_{\mathrm{B}}) & \overset{\cong}{\to} & Gr^W_{\dim +2l} H^{i+l}(\mathcal{M}_{\mathrm{B}}) \\ x & \mapsto & x \cup \alpha^l \end{array}$$

3. *(purity conjecture)*

$$W_k H^k(\mathcal{M}_{\mathrm{B}}) \overset{\tau^*_{RH}}{\cong} H^k(\mathcal{M}^*_{\mathrm{DR}})$$

4. *($P = W$)*
   perverse filtration P on $H^*(\mathcal{M}_{\mathrm{Dol}})$ induced by Hitchin map χ

$$W_{2k} H^*(\mathcal{M}_{\mathrm{B}}) = P_k H^*(\mathcal{M}_{\mathrm{Dol}})$$

- proved for $\mathrm{G} = \mathrm{GL}_2$ and many consintency checks

Toric hyperkähler varieties

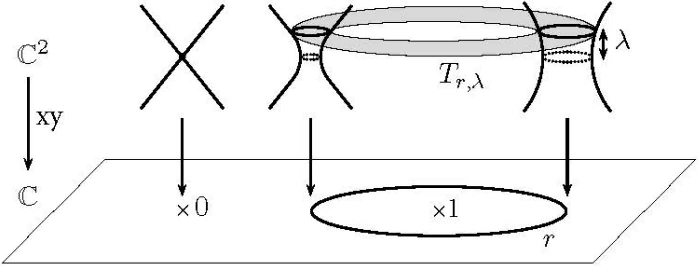
- Bielawski–Dancer (2000) Hausel–Sturmfels (2002)
- $A \in M_{dxn}(\mathbb{Z})$ surj. $\rightsquigarrow 0 \to \mathbb{Z}^{n-d} \overset{B}{\to} \mathbb{Z}^n \overset{A}{\to} \mathbb{Z}^d \to 0$
- taking Hom to $\mathbb{T} := \mathbb{C}^\times \rightsquigarrow 0 \leftarrow \mathbb{T}^{n-d} \overset{B^T}{\leftarrow} \mathbb{T}^n \overset{A^T}{\leftarrow} \mathbb{T}^d \to 0$
- $\mathbb{T}^d \subset \mathbb{T}^n \circlearrowright T^*\mathbb{C}^n$; moment map
$$\begin{array}{cccc} \nu_A : & (T^*\mathbb{C})^n & \to & (\mathfrak{t}^d)^* \\ & {\scriptstyle (x_i y_i)_i} \downarrow & & \wr\| \\ & \mathbb{C}^n & \overset{A}{\to} & \mathbb{C}^d \end{array}$$
- $Q_A^\xi := \nu_A^{-1}(\xi)//\mathbb{T}^d$ *toric hyperkähler variety* of dim $= 2(n-d)$
- $\mathbb{T}^{n-d} \circlearrowright Q_A^\xi$ with moment map $\nu : Q_A^\xi \to (\mathfrak{t}^{n-d})^*$ whose discriminental locus is a hyperplane arrangement $\mathcal{H}_A \subset (\mathfrak{t}^{n-d})^*$ modeled on $B^T = [b_1, \ldots, b_n] \in \mathbb{Z}^{n-d}$
- $H^*(Q_A^\xi)$ understood from the combinatorics of $\mathcal{H}_A$
  e.g. $\dim H^*(Q_A^\xi) = \#$ vertices of $\mathcal{H}_A$
- example: for any quiver Γ with n edges and $d+1$ vertices
  $\rightsquigarrow A_\Gamma(e_{ij}) = v_i - v_j$ a surjective matrix $A_\Gamma \in M_{d\times n}(\mathbb{Z})$
  $\rightsquigarrow Q_\Gamma^\xi := Q_{A_\Gamma}^\xi$ *toric quiver variety*

Toric character varieties

- Crawley-Boevey–Shaw (2006)
- $Z = \mathbb{C}^2 \setminus \{1 - xy = 0\}$ with symplectic form $\Omega = \frac{dx \wedge dy}{1-xy}$ and usual $\mathbb{T}$-action is quasi-Hamiltonian with moment map
$$\begin{array}{cccc} \mu : & Z & \to & \mathbb{T} \\ & (x,y) & \mapsto & 1 - xy \end{array}$$
- $\mathbb{T}^d \subset \mathbb{T}^n \circlearrowright Z^n$ with moment map
$$\begin{array}{cccc} \mu_A : & Z^n & \to & \mathbb{T}^d \\ & \mu^n \downarrow & & \wr\!\parallel \\ & \mathbb{T}^n & \xrightarrow{A} & \mathbb{T}^d \end{array}$$
- for generic $\zeta \in \mathbb{T}^d$ define $\mathcal{M}_{\mathrm{B}}^{\zeta} := \mu_A^{-1}(\zeta)//\mathbb{T}^d$ *toric Betti space* of dim $= 2(n-d)$ with symplectic form $\Omega \in \Omega^2(\mathcal{M}_{\mathrm{B}}^{\zeta})$ of Alexeev-Malkin-Meinrenken (1998)
- Γ quiver $\rightsquigarrow A = A_\Gamma \rightsquigarrow \mathcal{M}_{\mathrm{B}}^{\zeta}$ multiplicative quiver variety of Crawley-Boevey–Shaw (2006)

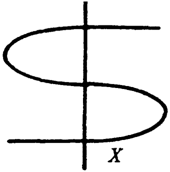
Special Lagrangian fibration on $\mathcal{M}_{\mathrm{B}}^{\zeta}$

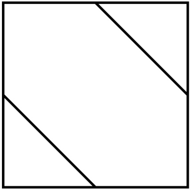
- Auroux (2009): $\chi : \begin{array}{ccc} Z & \to & \mathbb{R}^2 \\ (x,y) & & (log(|1-xy|), |x|^2-|y|^2) \end{array}$
  proper special Lagrangian fibration:
  $$\chi^{-1}(r,\lambda) = T_{r,\lambda} \cong \begin{cases} \mathbb{T}^2_{\mathbb{R}} \cong U(1)^2 & (r,\lambda) \neq (0,0) \\ \text{pinched torus} & (r,\lambda) = (0,0) \end{cases}$$



- $\rightsquigarrow \chi_A : \mathcal{M}_{\mathrm{B}}^{\zeta} \to (\mathbb{R}^2)^{n-d}$ proper special Lagrangian fibration;
  "toric Hitchin map in the Betti complex structure"
  degeneracy locus of χ_A is hyperplane arrangement $\mathcal{H}_A$ in $(\mathbb{R}^2)^{n-d}$ modelled on vector configuration $[b_1, \ldots, b_n] \in \mathbb{Z}^{n-d}$

Toroidal core

- $\zeta \in \mathbb{T}^d_{\mathbb{R}} \subset \mathbb{T}^d_{\mathbb{C}} \rightsquigarrow \mathcal{H}_A$ linear hyperplane arrangement $\rightsquigarrow \mathcal{C}^\zeta_A := \chi_A^{-1}(0)$ *toroidal core*: non-normal compact toric variety over a toroidal hyperplane arrangement
- Γ quiver $\rightsquigarrow \mathcal{C}^\zeta_{A_\Gamma} \cong_{diff} \overline{Jac}_\zeta(C_\Gamma)$ compactified Jacobian of reducible nodal rational curve C_Γ of Oda-Seshadri (1979)
- e.g. $C_\Gamma \cong$ 

$\overline{Jac}_\zeta(C_\Gamma) \cong$ 

Theorem (Hausel-Proudfoot 2015)

$\zeta \in \mathbb{T}^d_{\mathbb{R}} \subset \mathbb{T}^d_{\mathbb{C}}$ *generic* $\rightsquigarrow \mathcal{C}^\zeta_A \subset \mathcal{M}^\zeta_{\mathrm{B}}$ *is a homotopy equivalence*

- proof by Morse theory for $|\chi_A|^2 : \mathcal{M}^\zeta_{\mathrm{B}} \to \mathbb{R}$

Toric Hodge-Tate and Curious Hard Lefschetz

Theorem (Hausel–Proudfoot, 2015)

$H^*(\mathcal{M}_{\mathrm{B}}^{\zeta})$ is Hodge-Tate and satisfies Curious Hard Lefschetz.

- sketch of proof:
- define $Z^x := Z \setminus \{x = 0\} \cong \mathbb{T}^2$ "toric cluster torus"
- $S \subset \{1, \ldots, n\} \rightsquigarrow (\mathcal{M}_{\mathrm{B}}^{\zeta})_S := Z^S \times (Z^x)^{S^c} ////_{\zeta}^{qH} \mathbb{T}^d \subset \mathcal{M}_{\mathrm{B}}^{\zeta}$
- $b_S \subset \{b_1, \ldots, b_n\} \subset \mathbb{Z}^{n-d}$ linearly independent $\Rightarrow$

  $(\mathcal{M}_{\mathrm{B}}^{\zeta})_S \cong Z^{|S|} \times \mathbb{T}^{n-d-|S|}$ in particular satisfies HT and CHL
- $(\mathcal{M}_{\mathrm{B}}^{\zeta})_{S_1} \cap (\mathcal{M}_{\mathrm{B}}^{\zeta})_{S_2} = (\mathcal{M}_{\mathrm{B}}^{\zeta})_{S_1 \cap S_2}$
- claim: $\mathcal{M}_{\mathrm{B}}^{\zeta} = \bigcup_{b_S \text{ lin. ind.}} (\mathcal{M}_{\mathrm{B}}^{\zeta})_S$
- result follows from Mayer-Vietoris

Toric purity

Theorem (Hausel–Proudfoot, 2015)

$$W_k H^k(\mathcal{M}_{\mathrm{B}}^{e^\xi}) \cong H^k(Q_A^\xi)$$

- proof: define $\tau_{RH} : \mathbb{C}^2 \to Z$:

$$\begin{array}{ccc} (x,y) \in \mathbb{C}^2 & \xrightarrow{\tau_{RH}} & \begin{cases} \left(x, \frac{1-\exp(xy)}{x}\right) \in Z & x \neq 0 \\ (0,-y) \in Z & x = 0 \end{cases} \\ xy \downarrow & & \downarrow 1 - xy \\ \mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^\times \end{array}$$

- $\rightsquigarrow \tau_{RH} : Q_A^\xi \to \mathcal{M}_{\mathrm{B}}^{e^\xi}$
  $\rightsquigarrow \tau_{RH}^* : W_k H^k(\mathcal{M}_{\mathrm{B}}^{e^\xi}) \to H^k(Q_A^\xi)$ is surjective

- $\dim(W_* H^*(\mathcal{M}_{\mathrm{B}}^{e^\xi})) \overset{CHL}{=} \dim(H^{mid}(\mathcal{M}_{\mathrm{B}}^{e^\xi})) = \dim(H^{top}(\mathcal{C}_A^{e^\xi}))$
  $=$ #top dim regions in toroidal hyperplane arrangement
  $=$ #vertices of hyperplane arrangement $= \dim(H^*(Q_A^\xi))$ ∎

Comments on toric $P = W$

- recall $\chi : Z \to \mathbb{R}^2$
- $\chi^{-1}(\Delta) \cong_{diff} T$ where $T \to \Delta$ is the Tate curve
- $\rightsquigarrow$ a neighbourhood of $\mathcal{C}_A^\zeta \subset \mathcal{M}_{\mathrm{B}}^\zeta$ is diffeomorphic to a local abelian fibration with central singular fiber the toroidal core $\mathcal{C}_A^\zeta \rightsquigarrow$ perverse filtration on $H^*(\mathcal{C}_A^\zeta)$

Conjecture (de Cataldo-Hausel-Migliorini, 2007)

$\zeta \in \mathbb{T}_{\mathbb{R}}^d \subset \mathbb{T}_{\mathbb{C}}^d \rightsquigarrow W_{2k} H^*(\mathcal{M}_{\mathrm{B}}^\zeta) \cong P_k(H^*(\mathcal{C}_A^\zeta))$

- would follow from Mayer-Vietoris if we had $\mathcal{C}_A^\zeta := \chi_A^{-1}(\Delta_{bd}) \sim \mathcal{M}_{\mathrm{B}}^\zeta$ when $\zeta \in (\mathbb{R}^\times)^d \subset \mathbb{T}^d$ $\Delta_{bd} \subset \mathbb{R}^{n-d}$ bounded complex of the hyperplane arrangement

Problem

Can one cover the usual GL_n-character varieties $\mathcal{M}_{\mathrm{B}}$ with the (toric) character varieties corresponding to integral (nodal) spectral curves?