Toric Non-Abelian Hodge Theory

Project in progress with Nick Proudfoot

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Conference on Mathematical Physics and Geometric Analysis
January 14 - 17, 2008, Fields Institute, Toronto

Non-Abelian Hodge Theory

- Simpson (1990), Hitchin (1987) for Riemann surfaces
- G reductive complex algebraic group, M smooth complex projective variety
- Betti cohomology:

$$\mathcal{M}_{\mathrm{B}} := H^1_{\mathrm{B}}(M, \mathrm{G}) = \left\{ \begin{array}{c} \text{moduli space of representations} \\ \text{of } \pi_1(M) \to \mathrm{G} \end{array} \right\}$$

- De Rham cohomology:

$$\mathcal{M}_{\mathrm{DR}} := H^1_{\mathrm{DR}}(M, \mathrm{G}) = \{\text{moduli space of flat } \mathrm{G}\text{-connections on } M\}$$

- Delbeault cohomology:

$$\mathcal{M}_{\mathrm{Dol}} := H^1_{\mathrm{Dol}}(M, \mathrm{G}) = \{\text{moduli space of } \mathrm{G}\text{-Higgs bundles on } M\}$$

- Non-Abelian Hodge Theorem: $\mathcal{M}_{\mathrm{Dol}} \cong_{diff} \mathcal{M}_{\mathrm{DR}} \cong_{diff} \mathcal{M}_{\mathrm{B}}$

Geometric aspects of NAHT for Riemann surfaces

- $\mathrm{G} = \mathrm{GL}_n = \mathrm{GL}(n, \mathbb{C})$;
  $M = \Sigma_\mu$ coloured Riemann surface:
  - Σ compact Riemann surface with punctures
  - $a_1, \ldots, a_k \in \Sigma$ coloured by
  - $\mu = (\mu^1, \ldots, \mu^k) \in \mathcal{P}(n)^{\{1,\ldots,k\}}$ a partition of n at each puncture

- $\mathcal{M}^\mu_{\mathrm{Hit}} = \left\{ \begin{array}{c} \text{moduli space of solutions of} \\ \text{Hitchin self-duality equations on } \Sigma_\mu \end{array} \right\}$

  hyperkähler: $(\mathcal{M}^\mu_{\mathrm{Hit}}, I) \cong \mathcal{M}^\mu_{\mathrm{Dol}}$

  $(\mathcal{M}^\mu_{\mathrm{Hit}}, J) \cong (\mathcal{M}^\mu_{\mathrm{Hit}}, K) \cong \mathcal{M}^\mu_{\mathrm{DR}} \overset{RH}{\cong} \mathcal{M}^\mu_{\mathrm{B}}$

$$\begin{array}{rcl} RH : \mathcal{M}^\mu_{\mathrm{DR}} & \to & \mathcal{M}^\mu_{\mathrm{B}} \\ (E_\mu, \nabla) & \mapsto & \mathit{monodromy}(\nabla) \end{array}$$

Geometric aspects of NAHT for Riemann surfaces

- $\mathcal{M}^{\mu}_{\mathrm{Dol}} = \left\{ \begin{array}{c} \text{moduli space of } \mu\text{-parabolic rank } n \\ \text{Higgs bundles } (E_\mu, \phi) \text{ on } \Sigma \end{array} \right\}$

  the *Hitchin map*:

$$\begin{array}{rcl} \chi : \mathcal{M}^{\mu}_{\mathrm{Dol}} & \to & \mathcal{H}^{\mu} \\ (E_\mu, \phi) & \mapsto & CharPol(\phi) \end{array}$$

  is a completely integrable Hamiltonian system;

- $\mathbb{C}^{\times}$ acts on $\mathcal{M}^{\mu}_{\mathrm{Dol}}$ by $(E_\mu, \phi) \mapsto (E_\mu, \lambda\phi)$

  downward Morse flow $= \chi^{-1}(0)$, the *nilpotent cone* .

Geometric aspects of NAHT for Riemann surfaces

- $(\tilde{\mathcal{C}}_1, \ldots, \tilde{\mathcal{C}}_k)$ semisimple conjugacy classes in GL_n of type μ.

$$\mathcal{M}_{\mathrm{B}}^{\mu} := \left\{ \begin{array}{c} A_1, B_1, ..., A_g, B_g \in \mathrm{GL}_n, C_1 \in \tilde{\mathcal{C}}_1, ..., C_k \in \tilde{\mathcal{C}}_k \\ [A_1, B_1] \cdots [A_g, B_g] C_1 \cdots C_k = I_n \end{array} \right\} // \mathrm{GL}_n$$

- $(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ semisimple adjoint orbits in $\mathfrak{gl}_n$ of type μ.

$$\mathcal{M}_{\mathrm{DR}}^{\mu} := \left\{ \begin{array}{c} \text{moduli space of meromorphic rank } n \text{ flat connections} \\ \text{with simple poles at the punctures and residue in } \mathcal{C}_i \end{array} \right\}$$

When $\Sigma = \mathbb{P}^1$ a point in

$$\mathcal{Q}^{\mu} := \{ C_1 \in \mathcal{C}_1, \ldots, C_k \in \mathcal{C}_k | C_1 + \cdots + C_k = 0 \} // \mathrm{GL}_n$$

gives meromorphic flat connection $\sum_{i=1}^{k} C_i \frac{dz}{z - a_i} \in \mathcal{M}_{\mathrm{DR}}^{\mu}$ on $\mathbb{P}^1$

$$\Downarrow$$

$$\mathcal{Q}^{\mu} \quad \subset \quad \mathcal{M}_{\mathrm{DR}}^{\mu}$$

Cohomological aspects of NAHT for Σ_μ

- Morse theory for $\mathbb{C}^\times \circlearrowright \mathcal{M}^\mu_{\mathrm{Dol}}$ by $(E_\mu, \phi) \mapsto (E_\mu, \lambda\phi)$

$$\Downarrow$$

$$H^*(\mathcal{M}^\mu_{\mathrm{Dol}}) \cong H^*(\chi^{-1}(0)) \cong \bigoplus_{\cup F_i = (\mathcal{M}^\mu_{\mathrm{Dol}})^{\mathbb{C}^\times}} H^{*+\lambda_i}(F_i)$$

- Mixed Hodge structure is pure on $H^*(\mathcal{M}^\mu_{\mathrm{Dol}})$ and $H^*(\mathcal{M}^\mu_{\mathrm{DR}})$ but is not pure on $H^*(\mathcal{M}^\mu_{\mathrm{B}})$

Cohomological aspects of NAHT for Σ_μ

- *Purity Conjecture*:

$$\text{Pure part of } H^*(\mathcal{M}_{\mathrm{B}}^\mu) \cong H^*(\mathcal{Q}^\mu), \text{ if } \mu \text{ is indivisible.}$$
$$PH_c(\mathcal{M}_B^\mu, \sqrt{q}) = A_{\Gamma_\mu}(\mathbf{v}_\mu, q), \text{ if } \mu \text{ is divisible.}$$
$$\Downarrow$$
$$\text{Kac's conjecture for star-shaped quivers } \Gamma_\mu$$

- *Curious Poincaré Duality Conjecture:*

$$H^{p,p;k}(\mathcal{M}_{\mathrm{B}}^\mu) \cong H^{d_\mu - p, d_\mu - p; d_\mu + k - 2p}(\mathcal{M}_{\mathrm{B}}^\mu)$$
$$\Downarrow$$
$$PH^*(\mathcal{M}_B^\mu) \cong H^{d_\mu}(\mathcal{M}_B^\mu)$$

- *Master Conjecture* with Macdonald polynomials $\tilde{H}_\lambda(\mathbf{x}_i; q, t)$

$$\sum_{p,k} h^{p,p;k}(\mathcal{M}_{\mathrm{B}}^\mu) q^p t^k = (t\sqrt{q})^{d_\mu}(q-1)(1-\tfrac{1}{qt^2}) \cdot$$
$$\cdot \left\langle \mathrm{Log}\left(\sum_{\lambda\in\mathcal{P}} \left(\prod_{i=1}^k \tilde{H}_\lambda(\mathbf{x}_i; q, \tfrac{1}{qt^2})\right) \mathcal{H}_\lambda(q, \tfrac{1}{qt^2})\right), h_\mu \right\rangle$$

- the pure part and the $t = -1$ specialization of the Master Conjecture are theorems of (Hausel, Letellier, Villegas; 2007)

Aspects of NAHT for Σ^μ

$$
\begin{array}{cccccc}
 & & & (\mathcal{M}^\mu_{\mathrm{Hit}}, g) & & \\
 & & \swarrow & \downarrow & \searrow & \\
\chi^{-1}(0) \subset & \mathcal{M}^\mu_{\mathrm{Dol}} & \cong_{diff} & \mathcal{M}^\mu_{\mathrm{DR}} & \overset{RH}{\cong} & \mathcal{M}^\mu_{\mathrm{B}} \\
 & \chi \downarrow & & \rotatebox{90}{$\hookrightarrow$} & \nearrow & \\
 & \mathcal{H}^\mu & & \mathcal{Q}^\mu & &
\end{array}
$$

- Purity Conjecture: $PH^*(\mathcal{M}^\mu_{\mathrm{B}}) \cong H^*(\mathcal{Q}^\mu)$
- Curious Poincaré Duality: $H^{p,p;k}(\mathcal{M}^\mu_{\mathrm{B}}) \cong H^{d_\mu - p, d_\mu - p; d_\mu + k - 2p}(\mathcal{M}^\mu_{\mathrm{B}})$
- Master Conjecture: combinatorial formula for $\sum_{p,k} h^{p,p;k}(\mathcal{M}^\mu_{\mathrm{B}}) q^p t^k$

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 & \chi \downarrow & & \hookuparrow & \nearrow & & \\
 & \mathcal{H}^\mu & & \color{red}{\mathcal{Q}^\mu} & & &
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Quiver varieties

- Quiver $\Gamma = (\mathcal{V}, \mathcal{E})$, dimension vector $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_n) \in \mathbb{N}^{\mathcal{V}}$
  $\rightsquigarrow \mathcal{Q}^{\mathbf{v}} := T^*\mathrm{Rep}(\Gamma, \mathbf{v})////\mathrm{GL}_{\mathbf{v}}$
  a smooth hyperkähler *quiver variety*
- Recall

  $$\mathcal{Q}^{\mu} := \{C_1 \in \mathcal{C}_1, \ldots, C_k \in \mathcal{C}_k | C_1 + \cdots + C_k = 0\} //\mathrm{GL}_n$$

  Observation: $\mathcal{Q}^{\mu} \cong \mathcal{Q}^{\mathbf{v}_{\mu}}$ is the quiver variety associated to a certain star-shaped quiver Γ_{μ} and dimension vector $\mathbf{v}_{\mu}$
- [Crawley-Boevey–Shaw, 2006]: $(\Gamma, \mathbf{v}) \rightsquigarrow \mathcal{M}_B^{\mathbf{v}}$ *multiplicative quiver variety*, using group valued symplectic quotients of Alekseev-Malkin-Meinrenken.

  For a star-shaped quiver Γ_{μ}:

  $$\mathcal{M}_B^{\mathbf{v}_{\mu}} \cong \mathcal{M}_B^{\mu} = \left\{C_1 \in \tilde{\mathcal{C}}_1, \ldots, C_k \in \tilde{\mathcal{C}}_k | C_1 \cdots C_k = I\right\} //\mathrm{GL}_n$$

Graphical Non-Abelian Hodge Theory Wannabe for $(\Gamma, \mathbf{v})$:

$$
\begin{array}{ccccccccc}
 & & & & (\mathcal{M}^{\mathbf{v}}_{\mathrm{Hit}}, g) & & & \\
 & & & \swarrow & \downarrow & \searrow & & \\
\chi_{\mathbf{v}}^{-1}(0) \subset & \mathcal{M}^{\mathbf{v}}_{\mathrm{Dol}} & \cong_{diff} & & \mathcal{M}^{\mathbf{v}}_{\mathrm{DR}} & & \overset{RH}{\cong} & \mathcal{M}^{\mathbf{v}}_{\mathrm{B}} \\
 & \chi_{\mathbf{v}} \downarrow & & & \hookuparrow & & \nearrow & \\
 & \mathcal{H}^{\mathbf{v}} & & & \mathcal{Q}^{\mathbf{v}} & & &
\end{array}
$$

- Purity Conjecture: $PH^*(\mathcal{M}^{\mathbf{v}}_{\mathrm{B}}) \cong H^*(\mathcal{Q}^{\mathbf{v}})$
- Curious Poincaré Duality:
  $H^{p,p;k}(\mathcal{M}^{\mathbf{v}}_{\mathrm{B}}) \cong H^{d_{\mathbf{v}}-p,d_{\mathbf{v}}-p;d_{\mathbf{v}}+k-2p}(\mathcal{M}^{\mathbf{v}}_{\mathrm{B}})$
- Master Conjecture: combinatorial formula for
  $\sum_{p,k} h^{p,p;k}(\mathcal{M}^{\mathbf{v}}_{\mathrm{B}}) q^p t^k$

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- Purity Conjecture: $PH^*(\mathcal{M}^{\mathbf{v}}_{\mathrm{B}}) \cong H^*(\mathcal{Q}^{\mathbf{v}})$
- Curious Poincaré Duality:
  $H^{p,p;k}(\mathcal{M}^{\mathbf{v}}_{\mathrm{B}}) \cong H^{d_{\mathbf{v}}-p,d_{\mathbf{v}}-p;d_{\mathbf{v}}+k-2p}(\mathcal{M}^{\mathbf{v}}_{\mathrm{B}})$
- Master Conjecture: combinatorial formula for
  $\sum_{p,k} h^{p,p;k}(\mathcal{M}^{\mathbf{v}}_{\mathrm{B}}) q^p t^k$

Graphical Non-Abelian Hodge Theory would

- unify NAHT with the theory of quiver varieties ($\rightsquigarrow$ representation theory of Kac-Moody algebras, Nakajima)
- the Graphical Hitchin map $\chi_{\mathbf{v}} : \mathcal{M}^{\mathbf{v}}_{Dol} \to \mathcal{H}^{\mathbf{v}}$ would give many more examples of ACIHS
- the Graphical Purity conjecture $PH_c(\mathcal{M}^{\mathbf{v}}_B, \sqrt{q}) = A_\Gamma(\mathbf{v}_\mu, q)$ would imply Kac's conjecture from 1982, that the coefficients of $A_\Gamma(\mathbf{v}_\mu, q)$ are positive

Toric hyperkähler varieties

- [Bielawski-Dancer, 2000] differential geometric, [Hausel-Sturmfels, 2002] algebraic combinatorics approach
- Basic Hamiltonian action: $\mathbb{T} := \mathbb{C}^\times \circlearrowright \mathbb{C}^2$ $\lambda(z,w) = (\lambda z, \lambda^{-1} w)$ with moment map $\mu := \mathbb{C}^2 \to \mathbb{C}$ $\mu(z,w) = zw$
- $\mathcal{B}$ affine hyperplane arrangement in $\mathbb{Q}^d$, with $|\mathcal{B}| = n - d \rightsquigarrow$ $\mathbb{T}^{n-d}_{\mathcal{B}} \subset \mathbb{T}^n \circlearrowright (\mathbb{C}^2)^n$ with moment map $\mu_{\mathcal{B}} : (\mathbb{C}^2)^n \to \mathbb{C}^{n-d} \rightsquigarrow$ toric hyperkähler variety $\mathcal{Q}^{\mathcal{B}} := \mu_{\mathcal{B}}^{-1}(0)//_\theta \mathbb{T}^{n-d}$

  Example: $\Gamma \rightsquigarrow$ cographic $\mathcal{B}_\Gamma \rightsquigarrow \mathcal{Q}^{\mathcal{B}_\Gamma} \cong \mathcal{Q}^{\mathbf{1}}$ toric quiver variety
- $b_{2k}(\mathcal{Q}^{\mathcal{B}}) = h_k(M_{\mathcal{B}}) = \sum_{i=k}^{n-d} (-1)^{i-k} \binom{i}{k} f_i(\mathcal{B}_{bd})$
- middle Betti number $b_{2d}(\mathcal{Q}^{\mathcal{B}}) = f_d(\mathcal{B}_{bd})$ number of top dimensional bounded regions, Euler characteristic $\sum_k b_{2k}(\mathcal{Q}_{\mathcal{B}}) = f_0(\mathcal{B}_{bd})$ number of vertices

Toric Non-Abelian Hodge Theory for $\mathcal{B}$:

$$
\begin{array}{ccccccc}
 & & & & (\mathcal{M}^{\mathcal{B}}_{\mathrm{Hit}}, g) & & \\
 & & & \swarrow & \downarrow & \searrow & \\
\chi_{\mathcal{B}}^{-1}(0) \subset & \mathcal{M}^{\mathcal{B}}_{\mathrm{Dol}} & \cong_{diff} & & \mathcal{M}^{\mathcal{B}}_{\mathrm{DR}} & \overset{RH}{\cong} & \mathcal{M}^{\mathcal{B}}_{\mathrm{B}} \\
 & \chi_{\mathrm{B}} \downarrow & & & \uparrow & \nearrow & \\
 & \mathcal{H}^{\mathcal{B}} & & & \mathcal{Q}^{\mathcal{B}} & &
\end{array}
$$

- Purity Conjecture: $PH^*(\mathcal{M}^{\mathcal{B}}_{\mathrm{B}}) \cong H^*(\mathcal{Q}^{\mathcal{B}})$
- Curious Poincaré Duality:
  $H^{p,p;k}(\mathcal{M}^{\mathcal{B}}_{\mathrm{B}}) \cong H^{d-p,d-p;d+k-2p}(\mathcal{M}^{\mathcal{B}}_{\mathrm{B}})$
- Master Conjecture: combinatorial formula for
  $\sum_{p,k} h^{p,p;k}(\mathcal{M}^{\mathcal{B}}_{\mathrm{B}}) q^p t^k$

Construction of $\mathcal{M}_{\mathrm{B}}^{\mathcal{B}}$

- $Z := \mathbb{C}^2 \setminus \{zw + 1 = 0\}$
  basic action: $\mathbb{T} := \mathbb{C}^\times \circlearrowright Z$ $\lambda(z, w) = (\lambda z, \lambda^{-1} w)$
  holomorphic symplectic form: $\frac{dz \wedge dw}{1+zw}$
  group-valued moment map $\Phi := Z \to \mathbb{C}^\times$ $\Phi(z, w) = 1 + zw$
- $\mathcal{B}$, affine hyperplane arrangement in $\mathbb{Q}^d$, with $|\mathcal{B}| = n - d \rightsquigarrow$ $\mathbb{T}_{\mathcal{B}}^{n-d} \subset \mathbb{T}^n \circlearrowright Z^n$
  group-valued moment map $\Phi_{\mathcal{B}} : (Z)^n \to \mathbb{T}^{n-d}$
  $\rightsquigarrow$ *toric Betti space* $\mathcal{M}_{\mathrm{B}}^{\mathcal{B}} := \Phi_{\mathcal{B}}^{-1}(1) /\!/_\theta \mathbb{T}^{n-d}$

  Example: $\Gamma \rightsquigarrow$ cographic $\mathcal{B}_\Gamma \rightsquigarrow \mathcal{M}_{\mathrm{B}}^{\mathcal{B}_\Gamma} = \mathcal{M}_{\mathrm{B}}^{\mathbf{1}}$ toric multiplicative quiver variety of [Crawley-Boevey–Shaw, 2006]
- $E(\mathcal{M}_{\mathrm{B}}^{\mathcal{B}}, q) = \sum_{p,k} h^{p,p;k}(\mathcal{M}_{\mathrm{B}}^{\mathcal{B}}) q^p (-1)^k =$
  $\sum_i h_i(M_{\mathcal{B}})(q^2 - q + 1)^{d-i} q^i \Rightarrow E(\mathcal{M}_{\mathrm{B}}^{\mathcal{B}}, q) = q^d E(\mathcal{M}_{\mathrm{B}}^{\mathcal{B}}, 1/q)$
  $\Uparrow$
  Toric Curious Poincaré Duality

Toric Riemann-Hilbert map and Toric Purity Conjecture

- The local analytical isomorphism $RH_Z : \mathbb{C}^2 \to Z$:

$$
\begin{array}{ccc}
(z,w) \in \mathbb{C}^2 & \xrightarrow{RH_Z} & \begin{cases} \left(z, \frac{\exp(zw)-1}{z}\right) \in Z & z \neq 0 \\ (0,w) \in Z & z = 0 \end{cases} \\
zw \downarrow & & \downarrow 1+zw \\
\mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^\times
\end{array}
$$

- This induces a local analytical isomorphism $RH_{\mathcal{B}} : \mathcal{Q}^{\mathcal{B}} \to \mathcal{M}_{\mathrm{B}}^{\mathcal{B}}$
- For many $\mathcal{B}$ we can algebraically embed $i_{\mathcal{B}} : \mathcal{M}_{\mathrm{B}}^{\mathcal{B}} \hookrightarrow \mathcal{Q}^{\mathcal{B}}$
  $RH_{\mathcal{B}} \circ i_{\mathcal{B}}$ induces an isomorphism on $H^*(\mathcal{Q}_{\mathcal{B}})$.
  $\Rightarrow PH^*(\mathcal{M}_{\mathrm{B}}^{\mathcal{B}}) \cong H^*(\mathcal{Q}_{\mathcal{B}})$, the *Toric Purity Conjecture*

Construction of $\mathcal{M}_{\mathrm{Dol}}^{\mathcal{B}}$

- $\mathcal{B}_{A_\infty} = \{n \in \mathbb{R}^1 | n \in \mathbb{Z}\} \rightsquigarrow \mathcal{Q}^{\mathcal{B}_{A_\infty}}$ (aka A_∞ ALE space)
- $\mathbb{T} \circlearrowright \mathcal{Q}^{\mathcal{B}_{A_\infty}}$, moment map $\mu_{A_\infty} : \mathcal{Q}^{\mathcal{B}_{A_\infty}} \to \mathbb{C}$,
  $\mu_{A_\infty}^{-1}(0)$ infinite chain of $\mathbb{P}^1$'s, $\mu_{A_\infty}^{-1}(x) \cong \mathbb{C}^\times$, $x \neq 0$.
- $\mathbb{Z} \circlearrowright \mathcal{Q}^{\mathcal{B}_{A_\infty}}$ by shifting the chain of $\mathbb{P}^1$'s on $\mu_{A_\infty}^{-1}(0)$

  multiplying by x on $\mu_{A_\infty}^{-1}(x) \cong \mathbb{C}^\times$

  $T := \mu_{A_\infty}^{-1}(\Delta)/\mathbb{Z}$, $\Delta = \{|x| < 1\} \subset \mathbb{C}$,
- $\mathbb{T} \circlearrowright T$, moment map $\chi_T : T \to \Delta$.
  $\chi_T^{-1}(0) \cong$ nodal $\mathbb{P}^1$, $\chi_T^{-1}(x) \cong$ elliptic curve, $x \neq 0$
  $\rightsquigarrow$ *Tate curve*
- $\mathcal{B}$, affine hyperplane arrangement in $\mathbb{Q}^d$, with $|\mathcal{B}| = n - d$
  $\rightsquigarrow \mathbb{T}_{\mathcal{B}}^{n-d} \subset \mathbb{T}^n \circlearrowright T^n$ moment map $\mu_{\mathcal{B}} : T^n \to \mathbb{C}^{n-d}$
  $\rightsquigarrow$ *toric Dolbeault space* $\mathcal{M}_{\mathrm{Dol}}^{\mathcal{B}} := \mu_T^{-1}(0) /\!/_\theta \mathbb{T}^{n-d}$

Toric Hitchin map and Curious Poincaré Duality

- $\chi_T : T \to \Delta \rightsquigarrow \chi_{\mathcal{B}} : \mathcal{M}^{\mathcal{B}}_{\mathrm{Dol}} \to \Delta^d$ ACIHS *toric Hitchin map*
  $\chi_{\mathrm{B}}^{-1}(0)$ *toroidal core*: toric varieties glued together over the bounded regions in the toroidal hyperplane arrangement

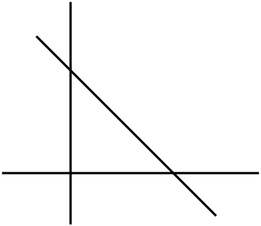
  $\tilde{\mathcal{B}} := (\mathcal{B} + \mathbb{Z}^d)/\mathbb{Z}^d \subset \mathbb{R}^d/\mathbb{Z}^d \cong \mathrm{U}(1)^d$
  $\Rightarrow b_{2d}(\mathcal{M}^{\mathcal{B}}_{\mathrm{Dol}}) = f_d(\tilde{\mathcal{B}})$
- simple combinatorial Morse theory on the toroidal hyperplane arrangement $\tilde{\mathcal{B}}$ implies that

  $f_d(\tilde{\mathcal{B}}) = \#\{$ top dimensional regions of $\tilde{\mathcal{B}}$ $\} = \#$ $\{$ vertices of $\tilde{\mathcal{B}}\}$

  $$\#\{\text{vertices of } \tilde{\mathcal{B}}\} = \#\{\text{vertices of } \mathcal{B}\} = f_0(\mathcal{B}_{bd}) = \sum_i b_{2i}(\mathcal{Q}^{\mathcal{B}})$$

  $\Rightarrow H^{2d}(\mathcal{M}^{\mathcal{B}}_{\mathrm{B}}) \cong H^*(\mathcal{Q}^{\mathcal{B}})$, a consequence of Curious Poincaré Duality, if we assume $\mathcal{M}^{\mathcal{B}}_{\mathrm{Dol}} \cong_{diff} \mathcal{M}^{\mathcal{B}}_{\mathrm{B}}$

Example: Calabi-$T^*\mathbb{P}^2$



- $\mathcal{B} \subset \mathbb{R}^2$:
- $\mathcal{Q}^{\mathcal{B}} \cong T^*\mathbb{P}^2 = \{\tilde{z}_1\tilde{w}_1 + \tilde{z}_2\tilde{w}_2 + \tilde{z}_3\tilde{w}_3 = 0\}//_\theta\mathbb{T}$

  $\mathcal{M}_{\mathrm{B}}^{\mathcal{B}} \cong mT^*\mathbb{P}^2 = \{(1+z_1w_1)(1+z_2w_2)(1+z_3w_3) = 1\}//_\theta\mathbb{T}$

  $\tilde{z}_i = z_i, \tilde{w}_1 = w_1, \tilde{w}_2 = (1+z_1w_1)w_2,$

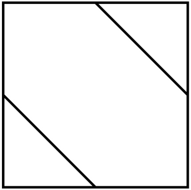
  $\tilde{w}_3 = (1+z_1w_1)(1+z_2w_2)w_3$

  $(1+z_1w_1)(1+z_2w_2)(1+z_3w_3) = 1 \Rightarrow \tilde{z}_1\tilde{w}_1 + \tilde{z}_2\tilde{w}_2 + \tilde{z}_3\tilde{w}_3 = 0$

  $\Downarrow$

  $\imath_{\mathcal{B}} : \mathcal{M}_{\mathrm{B}}^{\mathcal{B}} \to \mathcal{Q}^{\mathcal{B}}$ algebraic embedding $\rightsquigarrow PH^*(\mathcal{M}_{\mathrm{B}}^{\mathcal{B}}) \cong H^*(\mathcal{Q}^{\mathcal{B}})$

Example: Calabi-$T^*\mathbb{P}^2$



- $\tilde{\mathcal{B}} = (\mathcal{B} + \mathbb{Z}^2)/\mathbb{Z}^2$
- $\chi_{\mathcal{B}}^{-1}(0)$ is three toric varieties glued together according to the toroidal hyperplane arrangement
- $b_4(\mathcal{M}_{\mathrm{Dol}}^{\mathcal{B}}) = \#\{$ 2-dimensional regions in $\tilde{\mathcal{B}}\} = 3 =$ $\#\{$ vertices of $\tilde{\mathcal{B}}\} = b_0(T^*\mathbb{P}^2) + b_2(T^*\mathbb{P}^2) + b_4(T^*\mathbb{P}^2)$

  $$\Downarrow$$

  $H^4(mT^*\mathbb{P}^2) \cong H^*(T^*\mathbb{P}^2)$
- $\sum_{p,k} h^{p,p;k}(mT^*\mathbb{P}^2)q^p t^k =$ $1 + 2qt + qt^2 + 2q^2t^2 + q^2t^4 + q^3t^4 + 2q^3t^3 + q^4t^4$

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$$
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 & \chi \downarrow & & \hookuparrow & \nearrow & & \\
 & \mathcal{H}^{\mathcal{B}} & & \mathcal{Q}^{\mathcal{B}} & & &
\end{array}
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- Purity Conjecture: $PH^*(\mathcal{M}^{\mathcal{B}}_{\mathrm{B}}) \cong H^*(\mathcal{Q}^{\mathcal{B}})$
- Curious Poincaré Duality: $H^{p,p;k}(\mathcal{M}^{\mathcal{B}}_{\mathrm{B}}) \cong H^{d-p,d-p;d+k-2p}(\mathcal{M}^{\mathcal{B}}_{\mathrm{B}})$
- Master Conjecture: combinatorial formula for $\sum_{p,k} h^{p,p;k}(\mathcal{M}^{\mathcal{B}}_{\mathrm{B}}) q^p t^k$