Topology of the Hitchin map and the arithmetic of the character variety

Project in progress with Mark De Cataldo - Luca Migliorini

Tamás Hausel

Royal Society URF at University of Oxford
http://www.maths.ox.ac.uk/∼hausel/talks.html

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Original Problem

Problem

What is the space $\mathcal{H}^*$ of L^2 harmonic forms on $\mathcal{M}_{\mathrm{Dol}}$ the moduli space of Higgs bundles on a Riemann surface?

Motivation from physics

- (Sen 1994) conjecture predicting $\mathcal{H}^*$ for magnetic monopole moduli spaces
- (Vafa-Witten 1994) conjecture predicting $\mathcal{H}^*$ on the moduli of Yang-Mills instantons on ALE spaces

Hodge theory on complete Riemannian manifolds
$\Rightarrow$ have maps $H_c^* \to \mathcal{H}^* \to H^*$
$\Rightarrow$ $\mathrm{Im}(\mathrm{H_c^*} \to \mathrm{H^*})$ is a topological lower bound for $\mathcal{H}^*$

Problem

Determine the intersection form $H_c^{mid}(\mathcal{M}_{\mathrm{Dol}}) \to H^{mid}(\mathcal{M}_{\mathrm{Dol}})$.

Moduli of Higgs bundles

- Σ genus g Riemann surface
- $E \to \Sigma$ rank n holomorphic bundle, $\phi : E \to EK$ Higgs field
  (E, ϕ) Higgs bundle
- $\mathcal{M}_{\mathrm{Dol}}$ moduli space of rank n stable Higgs bundles of degree 1
- smooth, quasi-projective variety, with a natural hyperkähler metric
- $\mathbb{C}^\times$ acts on $\mathcal{M}_{\mathrm{Dol}}$ via $\lambda(E, \phi) = (E, \lambda\phi)$
- $Z = \mathcal{M}_{\mathrm{Dol}} //_\theta \mathbb{C}^\times$ highest level symplectic quotient - compact
- $\overline{\mathcal{M}}_{\mathrm{Dol}} = \mathcal{M}_{\mathrm{Dol}} \times \mathbb{C} //_\theta \mathbb{C}^\times = \mathcal{M}_{\mathrm{Dol}} \cup Z$ compactification by symplectic cutting (Hausel 1998)

Butterfly Lemma

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & H^{mid-2}(Z) & & & & \\
 & & & & \downarrow & \searrow^{L} & & & \\
0 & \rightarrow & H_c^{mid}(\mathcal{M}) & \rightarrow & H^{mid}(\overline{\mathcal{M}}) & \rightarrow & H^{mid}(Z) & \rightarrow & 0 \\
 & & & \searrow^{j} & \downarrow & & & & \\
 & & & & H^{mid}(\mathcal{M}) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

- where $L(\xi) = \xi \wedge \zeta$ is wedging with $\zeta = \eta_Z \in H^2(\overline{\mathcal{M}}) \cong H^2(Z)$
- "Butterfly lemma" $\Rightarrow \mathrm{coker(j)} \cong \mathrm{coker(L)}$
- "$\mathrm{coker(j)}$ measures how far is ζ from being ample" while "$\mathrm{coker(L)}$ measures the degeneracy of the intersection form" (Hausel 1998)

Character variety

(E,ϕ) Higgs bundle - stable $\mapsto$
(d_E,ϕ) unitary connection with Higgs field - solves Hitchin eqns $\mapsto$
$d_E+\phi+\phi^*$ complex connection- flat $\mapsto$
representation of $\pi_1(\Sigma)\to \mathrm{GL}_n(\mathbb{C})$

Non-Abelian Hodge theorem (Hitchin, Donaldson, Simpson, Corlette 1987-):

$$\mathcal{M}_{\mathrm{Dol}} \cong_{diff} \mathcal{M}_{\mathrm{B}}$$

where

$$\mathcal{M}_{\mathrm{B}} := Hom(\pi(\Sigma\backslash\{p\}) \to \mathrm{GL}(n,\mathbb{C})|\gamma_p \mapsto \exp(2\pi i/n)Id)//GL(n,\mathbb{C}) =$$

$$= \{A_1,B_1,\ldots,A_g,B_g \in GL(n,\mathbb{C})|$$
$$A_1^{-1}B_1^{-1}A_1B_1\ldots A_g^{-1}B_g^{-1}A_gB_g = \exp(2\pi i/n)Id\}//GL(n,\mathbb{C})$$

is a smooth affine variety

Weight filtration

- X variety defined over $\mathbb{Z}$
- $Frob_q : H^k(X(\overline{\mathbb{F}}_q); \mathbb{Q}_\ell) \to H^k(X(\overline{\mathbb{F}}_q); \mathbb{Q}_\ell)$ Frobenius automorphism
- (Deligne 1974) proved Weil's Riemann hypothesis: eigenvalues of $Frob_q$ have absolute value $q^{i/2}$ for $i \in \mathbb{N}$
- Jordan decomposition of $Frob_q$ on $H^k \Rightarrow$ weight filtration $W_l \subset H^k$ containing all Jordan blocks of eigenvalue with modulus $q^{i/2}$ $i \leq l$
- comparison theorem: $H^*(X(\mathbb{C}), \mathbb{C}) \cong H^*(X(\overline{\mathbb{F}}_q), \mathbb{Q}_\ell) \otimes \mathbb{C}$
- (Deligne 1972) proved the existence of $W_0 \subset \cdots \subset W_i \subset \cdots \subset W_{2k} = H^k(X, \mathbb{Q})$ for any complex algebraic variety X, which is
  - functorial
  - compatible with cup-product

Arithmetic of character variety

- For a smooth complex variety X define E-polynomial $E(X;q) = \sum W_i/W_{i-1}(H^k(X))(-1)^k q^{\frac{d-i}{2}}$
- (Katz 2008) proves that if $E(q) := |X(\mathbb{F}_q)|$ is polynomial in q then $E(X(\mathbb{C});q) = E(q)$
- (Hausel-Villegas 2008) calulates $E(\mathcal{M}_{\mathrm{B}};q) = |\mathcal{M}_B(\mathbb{F}_q)| = \sum_{\chi \in Irr(\mathrm{GL}_n(\mathbb{F}_q))} \frac{|\mathrm{GL}_n(\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-1}} \chi(\xi_n)$
- we find $E(\mathcal{M}_{\mathrm{B}};1/q) = q^d E(\mathcal{M}_B;q)$ palindromic by *Alvis-Curtis duality*

  $q^{\frac{n(n-1)}{2}} \chi(1)(1/q) = \chi'(1)(q)$ for dual pair $\chi, \chi' \in Irr(\mathrm{GL}_n(\mathbb{F}_q))$

- $\Rightarrow$ Curious Hard Lefschetz Conjecture:

$$\begin{array}{rccc} L^l: & Gr^W_{d-2l}(H^{i-l}(\mathcal{M}_{\mathrm{B}})) & \to & Gr^W_{d+2l} H^{i+l}(\mathcal{M}_{\mathrm{B}}) \\ & x & \mapsto & x \cup \alpha^l \end{array},$$

  where $\alpha \in W_4 H^2(\mathcal{M}_{\mathrm{B}})$

Perverse filtration

- $f : X \to Y$ a *proper* map between complex algebraic varieties of relative dimension d
- (de Cataldo-Migliorini 2005) introduce *perverse filtration* $\subset P_i \subset P_{i+1} \subset \dots P_k(X) \cong H^k(X)$ from the study of the Beilinson-Bernstein-Deligne-Gabber decomposition theorem for $f_!(\mathbb{Q}_X)$ into perverse sheaves
- the Relative Hard Lefschetz Theorem holds:

$$\begin{array}{cccc} L^l : & Gr^P_{d-l}(H^*(X)) & \to & Gr^P_{d+l}H^{*+2l}(X) \\ & x & \mapsto & x \cup \alpha^l \end{array},$$

where $\alpha \in H^2(X)$ is a relative ample class

Main conjecture

- recall Hitchin map $\begin{array}{rccc} \chi: & \mathcal{M}_{\mathrm{Dol}} & \to & \mathbb{A}^d \\ & (E,\phi) & \mapsto & \mathrm{charpol}(\phi) \end{array}$
- (Hitchin 1987) $\to$ completely integrable Hamiltonian system and *proper*

Conjecture

$P_k(\mathcal{M}_{\mathrm{Dol}}) \cong W_{2k}(\mathcal{M}_{\mathrm{B}})$ *under the isomorphism*
$H^*(\mathcal{M}_{\mathrm{Dol}}) \cong H^*(\mathcal{M}_{\mathrm{B}})$ *from non-Abelian Hodge theory*

- recipe (de Cataldo-Migliorini, $\geq$ 2008) for perverse filtration when X smooth and Y affine:
  take $Y_0 \subset \cdots \subset Y_i \subset \ldots Y_d = Y$
  s.t. Y_i generic with $\dim(Y_i) = i$ then

$$P_{k-i-1}H^j(X) = \mathrm{ker}(\mathrm{H^k(X) \to H^k(f^{-1}(Y_i))})$$

- thus Conjecture $\Rightarrow$ "topology of Hitchin map reflects the arithmetic of the character variety"

Universal Classes

- now on let $n = 2$, i.e. study rank 2 Higgs bundles
- $\mathbb{E} \to \mathcal{M}_{\mathrm{Dol}} \times \Sigma$ and $\Phi : \mathbb{E} \to \mathbb{E}K$, universal Higgs bundle $(\mathbb{E}, \Phi)|_{(E,\phi)\times\Sigma} = (E, \phi)$
- $$c_2(End(\mathbb{E})) = 2\alpha[\Sigma]^* + \sum_{i=1}^{2g} 4\psi e_i - \beta$$
  for some $\alpha \in H^2(\mathcal{M})$, $\psi_i \in H^3(\mathcal{M})$ and $\beta \in H^4(\mathcal{M})$.
- (Hausel-Villegas 2008) $\Rightarrow \alpha, \psi_i, \beta \in W_4$,
- Conjecture $\Rightarrow \alpha, \psi_i, \beta \in P_2 \Rightarrow$ $\psi_i, \beta \in \ker(\mathrm{H}^*(\mathcal{M}_{\mathrm{Dol}}) \to \mathrm{H}^*(\chi^{-1}(\mathrm{Y}_0)))$
- Yes! was proved by (Thaddeus 1990)
- $\beta \in P_2 H^4(\mathcal{M}_{\mathrm{Dol}})$ would mean $\beta \in \ker(\mathrm{H}^*(\mathcal{M}_{\mathrm{Dol}}) \to \mathrm{H}^*(\chi^{-1}(\mathrm{Y}_1)))$ i.e. β vanishes over a generic curve in $\mathbb{A}^d$.

Perverse filtration on $\overline{\mathcal{M}}_{\mathrm{Dol}}$

- $f : X \to Y$ with X, Y smooth and projective
  $\theta \in H^2(Y)$ ample $\zeta = f^{-1}(\theta)$ let
  $$\begin{array}{cccc} L_\zeta : & H^*(X) & \to & H^{*+2}(X) \\ & x & \mapsto & x \cup \zeta \end{array}$$
  (de Cataldo-Migliorini 2005)$\Rightarrow$
  $P_{l-k}H^l(X) = \sum_{i+j=d_Y-k} \mathrm{ker}(\mathrm{L}_\zeta^{\mathrm{i+1}}) \cap \mathrm{im}(\mathrm{L}_\zeta^{\mathrm{j}})$
- $$\begin{array}{cccc} L_\zeta^l : & Gr^P_{b+d}(H^{d_X+b-l}(X)) & \stackrel{\cong}{\to} & Gr^P_{b+d}H^{d_x+b+l}(X) \\ & x & \mapsto & x \cup \zeta^l \end{array},$$
- "It is as if the perverse filtration were calibrated precisely for the purpose of correcting the failure of the Hard Lefschetz Theorem for L_ζ" (de Cataldo-Migliorini 2005)
- recall $\overline{\mathcal{M}}_{\mathrm{Dol}} = \mathcal{M} \cup Z$,the Hitchin map also compactifies $\overline{\chi} : \overline{\mathcal{M}}_{\mathrm{Dol}} \to \overline{\mathbb{A}}^d$ in this case $\zeta = f^{-1}(\theta) = \eta_Z$
- our Butterfly Lemma studied the perverse filtration on $H^*(\overline{\mathcal{M}}_{\mathrm{Dol}})$!

Equivariant intersection numbers

- let X be a manifold with a $U(1)$ action such that $F = X^{U(1)}$ is compact $\Rightarrow$ *circle compact*
- for $\eta \in H^*_{U(1)}(X)$ define

$$\int_X \eta = \int_F \frac{\eta}{E(N_F)} \in \hat{H}^*_{U(1)}(pt) \cong \mathbb{Q}(u)$$

- we have (Hausel-Proudfoot 2004) Poincaré duality: i.e. the pairing $\int_X \eta \wedge \eta'$ is non-degenerate
- have equivariant universal classes $\alpha, \psi_i, \beta \in H^*_{U(1)}(\mathcal{M}_{\mathrm{Dol}})$ generating $H^*_{U(1)}(\mathcal{M}_{\mathrm{Dol}})$
- thus to determine $H^*_{U(1)}(\mathcal{M}_{\mathrm{Dol}})$ it is enough to evaluate $\int_{\mathcal{M}_{\mathrm{Dol}}} \alpha^k \beta^l \psi_{i_1} \dots \psi_{i_r}$
- with $\gamma = \sum \psi_i \wedge \psi_{i+g} \in H^6_{U(1)}(\mathcal{M}_{\mathrm{Dol}})$ enough to evaluate $\int_{\mathcal{M}_{\mathrm{Dol}}} \alpha^k \beta^l \gamma^m$

Bethe Ansatz residue formula

Conjectured by physicists (Moore-Nekrasov-Shatashivili 2000)

Theorem (Hausel-Szenes $\geq$ 2008)

For rank 2 Higgs bundles

$$\int_{\mathcal{M}_{\mathrm{Dol}}} \frac{\exp(T\alpha)}{(1-S\beta)(1-Q\gamma)} = (\mathrm{Res}_{\mathrm{y=0}} + \mathrm{Res}_{\mathrm{y=u}} + \mathrm{Res}_{\mathrm{y=-u}})\mathrm{BA},$$

where

$$BA = \frac{\left(\frac{1}{u-y} + \frac{1}{u+y} + T + \frac{Qy^2}{2}\right)^g}{(1-Sy^2)u^{g-1}y^{2g-2}(u^2-y^2)^{g-1}\left(\exp(Ty)\frac{u+y}{u-y} - \exp(-Ty)\frac{u-y}{u+y}\right)}$$

Kalkman residue formula

- $U(1)$ acts on smooth symplectic manifold X, with proper moment map $\mu : X \to \mathbb{R}_{\geq 0}$ so that symplectic quotient $Z = X//_\theta U(1) = \mu^{-1}(\theta)/U(1)$ is smooth compact
- $\eta \in H^{d_Z}_{U(1)}(X)$ gives $k(\eta) \in H^{d_Z}(Z) \cong H^{d_Z}_{U(1)}(\mu^{-1}(\theta))$
- then (Kalkman 1993)

$$\int_Z k(\eta) = \mathrm{Res}_{u=0} \sum_{\mu(F)<\theta} \int_F \frac{\eta}{E(N_F)},$$

where $\int_F \frac{\eta}{E(N_F)} \in \hat{H}^*_{U(1)}(pt) \cong \mathbb{Q}(u)$

- to conclude take $X = \mathcal{M}_{\mathrm{Dol}}$,using Bethe Ansatz residue formula+ Kalkman residue formula (Hausel-Szenes $\geq$ 2008) $\Rightarrow k(u)^{d-2}(k(\beta) + \frac{2}{3g-3} k(u)k(\alpha)) = 0$ on Z; thus $\zeta^{d-1}(\beta + \frac{2}{3g-3}\zeta\alpha) = 0$ on $\overline{\mathcal{M}}_{\mathrm{Dol}}$ therefore $\beta + \frac{2}{3g-3}\zeta\alpha \in P_2H^4(\overline{\mathcal{M}}_{\mathrm{Dol}})$ therefore $\beta \in P_2H^4(\mathcal{M}_{\mathrm{Dol}})$

Conclusion

- $\beta \in P_2 H^4(\mathcal{M}_{\mathrm{Dol}}) \Rightarrow \beta$ vanishes over a generic curve in $\mathbb{A}^d$
- $H^*(\mathcal{M}_{\mathrm{Dol}})$ generated by α, ψ_i, β
- thus $\alpha, \psi_i, \beta \in P_2$ mirroring $\alpha, \psi_i, \beta \in W_4$
- the weight filtration is compatible with cup product $\Rightarrow$ weight filtration on $H^*(\mathcal{M}_{\mathrm{B}})$ is determined
- our main Conjecture $\Rightarrow$ the perverse filtration of the Hitchin map is compatible with cup-product! Why?
- (Ngo 2008) also studies the perverse filtration arithmetically on $\mathcal{M}_{\mathrm{Dol}}$ to get the fundamental Langlands lemma
- (Frenkel-Witten 2007) study the topology of the Hitchin map over generic curves in connection with the Geometric Langlands program and mirror symmetry
- Any connection to our considerations?