Topology of the Hitchin map and the arithmetic of the character variety

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Hard Lefschetz Theorem

- Hodge theorem for compact oriented Riemannian manifold M: $H^*(M) \cong \mathcal{H}^*(M)$ the space of L^2 harmonic forms
- If M is additionally Kähler we have the action of an $N = 2$ supersymmetry algebra on $\mathcal{H}^*(M) \cong H^*(M) \Rightarrow$ yielding an action of $\mathrm{SU}(2)$ on $\mathcal{H}^*(M) \Rightarrow$ Hard Lefschetz Theorem:

$$L^l: \quad \begin{array}{ccc} H^{d_M - l}(M) & \stackrel{\cong}{\to} & H^{d_M + l}(M) \\ x & \mapsto & x \cup \alpha^l \end{array}$$

  where $\alpha \in H^2(M)$ is the Kähler class.
- when M is compact hyperkähler Hodge theory gives an action of an $N = 4$ supersymmetry algebra on $\mathcal{H}^*(M)$, which yields an action of $\mathrm{SO}(5)$ on $\mathcal{H}^*(M)$
- What happens when M is non-compact but complete?

Original Problem

Problem

What is the space $\mathcal{H}^*$ of L^2 harmonic forms on $\mathcal{M}_{\mathrm{Dol}}$ the moduli space of Higgs bundles on a Riemann surface?

Motivation from physics

- (Sen 1994) conjecture predicts $\mathcal{H}^*$ for magnetic monopole moduli spaces
- (Vafa-Witten 1994) conjecture predicts $\mathcal{H}^*$ on the moduli of Yang-Mills instantons on ALE spaces

Hodge theory on complete Riemannian manifolds
$\Rightarrow$ have maps $H^*_c \to \mathcal{H}^* \to H^*$
$\Rightarrow \mathrm{Im}(\mathrm{H}^*_\mathrm{c} \to \mathrm{H}^*)$ is a topological lower bound for $\mathcal{H}^*$

Theorem (Hausel 1998)

The intersection form $H^{mid}_c(\mathcal{M}_{\mathrm{Dol}}) \to H^{mid}(\mathcal{M}_{\mathrm{Dol}})$ is trivial for rank two Higgs bundles. (Thus $\mathcal{H}^*(\mathcal{M}_{\mathrm{Dol}})$ maybe trivial.)

Moduli of Higgs bundles

- Σ genus g Riemann surface
- $E \to \Sigma$ rank n holomorphic bundle, $\phi : E \to EK$ Higgs field
  (E, ϕ) Higgs bundle
- $\mathcal{M}_{\mathrm{Dol}}$ moduli space of rank n stable Higgs bundles of degree 1
- smooth, quasi-projective variety, with a natural hyperkähler metric
- (E, ϕ) Higgs bundle - stable $\mapsto$
  (d_E, ϕ) unitary connection with Higgs field - solves Hitchin eqns $\mapsto$
  $d_E + \phi + \phi^*$ complex connection- flat $\mapsto$
  representation of $\pi_1(\Sigma) \to \mathrm{GL}_n(\mathbb{C})$

Character variety

Non-Abelian Hodge theorem (Hitchin, Donaldson, Simpson, Corlette 1987-):

$$\mathcal{M}_{\mathrm{Dol}} \cong_{diff} \mathcal{M}_{\mathrm{B}}$$

where

$$\mathcal{M}_{\mathrm{B}} := Hom(\pi(\Sigma \backslash \{p\}) \to \mathrm{GL}(n,\mathbb{C}) | \gamma_p \mapsto \exp(2\pi i/n) Id) // GL(n,\mathbb{C}) =$$

$$= \{A_1, B_1, \ldots, A_g, B_g \in GL(n,\mathbb{C}) | \\ A_1^{-1} B_1^{-1} A_1 B_1 \ldots A_g^{-1} B_g^{-1} A_g B_g = \exp(2\pi i/n) Id\} // GL(n,\mathbb{C})$$

is a smooth affine variety

Weight filtration

- X variety defined over $\mathbb{Z}$
- $Frob_q : H^k(X(\overline{\mathbb{F}}_q); \mathbb{Q}_\ell) \to H^k(X(\overline{\mathbb{F}}_q); \mathbb{Q}_\ell)$ Frobenius automorphism
- (Deligne 1974) proved Weil's Riemann hypothesis: eigenvalues of $Frob_q$ have absolute value $q^{i/2}$ for $i \in \mathbb{N}$
- Jordan decomposition of $Frob_q$ on $H^k \Rightarrow$ weight filtration $W_l \subset H^k$ containing all Jordan blocks of eigenvalue with modulus $q^{i/2}$ $i \leq l$
- comparison theorem: $H^*(X(\mathbb{C}); \mathbb{C}) \cong H^*(X(\overline{\mathbb{F}}_q), \mathbb{Q}_\ell) \otimes \mathbb{C}$
- (Deligne 1972) proved the existence of $W_0 \subset \cdots \subset W_i \subset \cdots \subset W_{2k} = H^k(X; \mathbb{Q})$ for any complex algebraic variety X, which is
  - functorial
  - compatible with cup-product

Example

- Take $X = \mathbb{C}^\times = \mathbb{C} \setminus \{0\} \cong \{(x, y) \in \mathbb{C}^2 | xy = 1\}$
  $H^0(X; \mathbb{C}) \cong H^2_c(X; \mathbb{C}) \cong \mathbb{C}$, $H^1(X, \mathbb{C}) \cong H^1_c(X, \mathbb{C}) \cong \mathbb{C}$
- $X(\overline{\mathbb{F}}_q) = \overline{\mathbb{F}}_q^\times$
- $$\begin{array}{rccc} Frob_q & : \overline{\mathbb{F}}_q^\times & \to & \overline{\mathbb{F}}_q^\times \\ & x & \mapsto & x^q \end{array}$$
- $X(\overline{\mathbb{F}_q})^{Frob_q} = X(\mathbb{F}_q) = \mathbb{F}_q \setminus \{0\}$, thus $|X(\overline{\mathbb{F}_q})^{Frob_q}| = q - 1$
- Grothendieck-Lefschetz$\Rightarrow$ $|X(\overline{\mathbb{F}_q})^{Frob_q}| =$
  $\sum_{i=0}^2 (-1)^i \mathrm{Trace}(\mathrm{Frob_q} : \mathrm{H^i_c}(\mathrm{X}, \mathbb{Q}_\ell) \to \mathrm{H^i_c}(\mathrm{X}, \mathbb{Q}_\ell))$
- thus $1 = Frob_q : H^1_c(X; \mathbb{Q}_\ell) \to H^1_c(X, \mathbb{Q}_\ell))$ and
  $q \cdot \_ = Frob_q : H^2_c(X; \mathbb{Q}_\ell) \to H^2_c(X, \mathbb{Q}_\ell))$
- $\Rightarrow$
  $0 = W_1(H^1(X(\mathbb{C}), \mathbb{Q})) \subset W_2(H^1(X(\mathbb{C}); \mathbb{Q})) = H^1(X(\mathbb{C}), \mathbb{Q})$

Arithmetic of character variety

- For a smooth complex variety X define E-polynomial $E(X;q) = \sum W_i/W_{i-1}(H^k(X))(-1)^k q^{\frac{d-i}{2}}$
- (Katz 2008) proves that if $E(q) := |X(\mathbb{F}_q)|$ is polynomial in q then $E(X(\mathbb{C});q) = E(q)$
- (Hausel-Villegas 2008) calulates $E(\mathcal{M}_{\mathrm{B}};q) = |\mathcal{M}_B(\mathbb{F}_q)| = \sum_{\chi\in Irr(\mathrm{GL}_n(\mathbb{F}_q))} \frac{|\mathrm{GL}_n(\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-1}}\chi(\xi_n)$
- we find $E(\mathcal{M}_{\mathrm{B}};1/q) = q^d E(\mathcal{M}_B;q)$ palindromic by *Alvis-Curtis duality*

  $$q^{\frac{n(n-1)}{2}}\chi(1)(1/q) = \chi'(1)(q) \text{ for dual pair } \chi,\chi' \in Irr(\mathrm{GL}_n(\mathbb{F}_q))$$

- $\Rightarrow$ Curious Hard Lefschetz Conjecture:

  $$\begin{array}{cccc} L^l: & Gr^W_{d-2l}(H^{i-l}(\mathcal{M}_{\mathrm{B}})) & \to & Gr^W_{d+2l}H^{i+l}(\mathcal{M}_{\mathrm{B}}) \\ & x & \mapsto & x\cup\alpha^l \end{array},$$

  where $\alpha \in W_4H^2(\mathcal{M}_{\mathrm{B}})$

Perverse filtration

- $f : X \to Y$ a *proper* map between complex algebraic varieties of relative dimension d
- (de Cataldo-Migliorini 2005) introduce *perverse filtration* $\subset P_i \subset P_{i+1} \subset \dots P_k(X) \cong H^k(X)$ from the study of the Beilinson-Bernstein-Deligne-Gabber decomposition theorem for $f_!(\mathbb{Q}_X)$ into perverse sheaves
- the Relative Hard Lefschetz Theorem holds:

$$\begin{array}{rccc} L^l: & Gr^P_{d-l}(H^*(X)) & \to & Gr^P_{d+l}H^{*+2l}(X) \\ & x & \mapsto & x \cup \alpha^l \end{array}$$

  where $\alpha \in H^2(X)$ is a relative ample class

Main conjecture

- recall Hitchin map $\begin{array}{cccc} \chi: & \mathcal{M}_{\mathrm{Dol}} & \to & \mathbb{A}^d \\ & (E,\phi) & \mapsto & \mathrm{charpol}(\phi) \end{array}$
- (Hitchin 1987) $\to$ completely integrable Hamiltonian system and *proper*

Conjecture

$P_k(\mathcal{M}_{\mathrm{Dol}}) \cong W_{2k}(\mathcal{M}_{\mathrm{B}})$ *under the isomorphism*
$H^*(\mathcal{M}_{\mathrm{Dol}}) \cong H^*(\mathcal{M}_{\mathrm{B}})$ *from non-Abelian Hodge theory*

- recipe (de Cataldo-Migliorini, $\geq$ 2008) for perverse filtration when X smooth and Y affine:
  take $Y_0 \subset \cdots \subset Y_i \subset \ldots Y_d = Y$
  s.t. Y_i generic with $\dim(Y_i) = i$ then

$$P_{k-i-1}H^k(X) = \mathrm{ker}(\mathrm{H^k(X) \to H^k(f^{-1}(Y_i)))}$$

- thus Conjecture $\Rightarrow$ "topology of Hitchin map reflects the arithmetic of the character variety"

Universal Classes

- now on let $n = 2$, i.e. study rank 2 Higgs bundles
- $\mathbb{E} \to \mathcal{M}_{\mathrm{Dol}} \times \Sigma$ and $\Phi : \mathbb{E} \to \mathbb{E}K$, universal Higgs bundle $(\mathbb{E}, \Phi)|_{(E,\phi)\times\Sigma} = (E, \phi)$
- $$c_2(End(\mathbb{E})) = 2\alpha[\Sigma]^* + \sum_{i=1}^{2g} 4\psi_i e_i - \beta$$
  for some $\alpha \in H^2(\mathcal{M})$, $\psi_i \in H^3(\mathcal{M})$ and $\beta \in H^4(\mathcal{M})$.
- (Hausel-Villegas 2008) $\Rightarrow \alpha, \psi_i, \beta \in W_4$,
- Conjecture $\Rightarrow \alpha, \psi_i, \beta \in P_2 \Rightarrow$ $\psi_i, \beta \in \ker(\mathrm{H}^*(\mathcal{M}_{\mathrm{Dol}}) \to \mathrm{H}^*(\chi^{-1}(\mathrm{Y}_0)))$
- Yes! was proved by (Thaddeus 1990)
- $\beta \in P_2 H^4(\mathcal{M}_{\mathrm{Dol}})$ would mean $\beta \in \ker(\mathrm{H}^4(\mathcal{M}_{\mathrm{Dol}}) \to \mathrm{H}^4(\chi^{-1}(\mathrm{Y}_1)))$ i.e. β vanishes over a generic curve in $\mathbb{A}^d$.

Hitchin's heuristics for $\beta \in P_2 H^4(\mathcal{M}_{\mathrm{Dol}})$

- need $\beta \in \ker(\mathrm{H}^4(\mathcal{M}_{\mathrm{Dol}}) \to \mathrm{H}^4(\chi^{-1}(\mathrm{Y}_1)))$
- Around a singular fibre $\chi^{-1}(Y_1))$ topologically looks like $E \times T$ where
  - $T = T^{8g-8}$ is the Jacobian of the normalization of the singular spectral curve
  - $\tau : E \to \Delta \subset \mathbb{C}$ is an elliptic fibration the Tate curve: $\tau^{-1}\lambda \cong T^2$ for $\lambda \neq 0$ and $\tau^{-1}(0) =$ nodal $\mathbb{P}^1$
  - "*focus-focus singularity*"
- calculate $c_2(\mathcal{M}_{\mathrm{Dol}}) = (2g-2)\beta$
- note $c_2(\mathcal{M}_{\mathrm{Dol}})$ is a multiple of $\eta_{\mathcal{S}} \in H^4(\mathcal{M}_{\mathrm{Dol}})$, where $\mathcal{S} = \{x \in \mathcal{M}_{\mathrm{Dol}} | T\chi_x \text{ is not surjective}\}$ the singular locus of χ
- note $\mathcal{S} \cap E \times T = s \times T$
  where s is the singular point on the nodal $\mathbb{P}^1 = \tau^{-1}(0) \subset E$
- thus $\eta_{\mathcal{S}}$ can be moved to infinity on $\chi^{-1}(Y_1)$
- $\beta \in \ker(\mathrm{H}^4(\mathcal{M}_{\mathrm{Dol}}) \to \mathrm{H}^4(\chi^{-1}(\mathrm{Y}_1)))$

Conclusion

- $\beta \in P_2 H^4(\mathcal{M}_{\mathrm{Dol}}) \Rightarrow \beta$ vanishes over a generic curve in $\mathbb{A}^d$
- $H^*(\mathcal{M}_{\mathrm{Dol}})$ generated by α, ψ_i, β
- thus $\alpha, \psi_i, \beta \in P_2$ mirroring $\alpha, \psi_i, \beta \in W_4$
- the weight filtration is compatible with cup product $\Rightarrow$ weight filtration on $H^*(\mathcal{M}_{\mathrm{Dol}})$ is determined
- our main Conjecture $\Rightarrow$ the perverse filtration of the Hitchin map is compatible with cup-product! Why?
- (Ngo 2008) also studies the perverse filtration arithmetically on $\mathcal{M}_{\mathrm{Dol}}$ to get the fundamental Langlands lemma
- (Frenkel-Witten 2007) study the topology of the Hitchin map over generic curves in connection with the Geometric Langlands program and mirror symmetry
- Any connection to our considerations?