Topology and arithmetic of character varieties

Tamás Hausel

Chair of Geometry, EPF Lausanne
http://geom.epfl.ch/Hausel/talks/pdf

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GL_n character variety

- C genus g compact Riemann surface; fix $r > 0$ and $(d, r) = 1$

$$\mathcal{M}^r := \{A_1, B_1, .., A_g, B_g \in \mathrm{GL}_r | \prod_{i=1}^{g} A_i^{-1} B_i^{-1} A_i B_i = e^{\frac{2\pi i d}{r}} Id\} // \mathrm{PGL}_r$$

- non-singular affine
- $r = 1 \rightsquigarrow \mathcal{M}^1 \cong \mathrm{Hom}(\pi_1(C), \mathrm{GL}_1(\mathbb{C})) \cong (\mathbb{C}^\times)^{2g}$
- $r > 1$ and $g = 0 \rightsquigarrow \mathcal{M}^r = \varnothing$
- $r > 1$ and $g = 1 \rightsquigarrow$ Stone-von Neumann $\rightsquigarrow \mathcal{M}^r \cong (\mathbb{C}^*)^2$

Mixed Hodge polynomials

- (Deligne 1971) proved the existence of $W_0 \subset \cdots \subset W_i \subset \cdots \subset W_{2k} = H^k(X;\mathbb{Q})$ for any complex algebraic variety X which is
  - functorial
  - compatible with cup-product
- $H(X;q,t) = \sum \dim(W_i/W_{i-1}(H^k(X)))t^k q^{\frac{i}{2}}$, *mixed Hodge polynomial*
- $P(X;t) = H(X;1,t)$, *Poincaré polynomial*
- $E(X;q) = q^D H(X;1/q,-1)$, *E-polynomial* of X.

Theorem (Katz 2008)

If M is a smooth quasi-projective variety defined over $\mathbb{Z}$ and

$$\#\{M(\mathbb{F}_q)\} = E(q)$$

is a polynomial in q, then $E(M;q) = E(q)$.

Examples for mixed Hodge polynomials

- M smooth semi-projective variety $\rightsquigarrow$
  $W_{k-1}(H^k(M)) = 0$ because it is smooth
  $W_k(H^k(M)) = W_k(H^k(\mathcal{C})) = H^k(\mathcal{C}) = H^k(M) \Leftarrow \mathcal{C}$ projective
  such cohomology is called *pure*

  $\rightsquigarrow H(M; q, t) = P(M; tq^{\frac{1}{2}})$

  $\rightsquigarrow E(M; q) = q^D P(M; -q^{-\frac{1}{2}})$

- $H(T^*J^d(C); q, t) = H(J^d(C); q, t) = (1 + q^{\frac{1}{2}}t)^{2g}$ MHS is pure

  $E(T^*J^d(C); q) = q^g(1 - q^{\frac{1}{2}})^{2g}$ *not* polynomial count

- $\#(\mathrm{GL}_1(\mathbb{F}_q)) = q - 1 \overset{Katz}{\Rightarrow} E(\mathrm{GL}_1(\mathbb{C}); q) = E(\mathbb{C}^\times; q) = q - 1 \rightsquigarrow$
  $W_1 H^1(\mathbb{C}^\times) = 0 \rightsquigarrow H(\mathbb{C}^\times; q, t) = 1 + qt$ MHS *not* pure

- $\rightsquigarrow H((\mathbb{C}^\times)^{2g}; q, t) = (1 + qt)^{2g}$ and $E((\mathbb{C}^\times)^{2g}; q) = (q - 1)^{2g}$

- $E((\mathbb{C}^\times)^{2g}; q)$ palindromic $\Leftrightarrow E((\mathbb{C}^\times)^{2g}; q) = q^{2g} E((\mathbb{C}^\times)^{2g}; 1/q)$

- extends to $H((\mathbb{C}^\times)^{2g}; q, t) = (qt)^{2g} H((\mathbb{C}^\times)^{2g}; 1/qt^2, t)$

Mixed Hodge polynomials

Theorem (Frobenius 1896, Hurwitz 1902, Freed-Quinn 1993,...)

Let $z \in \mathrm{G}$ in a finite group G then

$$\#\{a_j, b_j \in \mathrm{G} | [a_1, b_1] \dots [a_g, b_g] = z\} =$$

$$= \sum_{\chi \in \mathrm{Irr}(\mathrm{G})} \frac{|\mathrm{G}|^{2g-1}}{\chi(1)^{2g-2}} \frac{\chi(z)}{\chi(1)}$$

- (Hausel-Villegas 2008) calculates $E(\mathcal{M}^r; q) \overset{Katz}{=} |\mathcal{M}^r(\mathbb{F}_q)| = \sum_{\chi \in Irr(\mathrm{GL}_r(\mathbb{F}_q))} \frac{|\mathrm{GL}_r(\mathbb{F}_q)|^{2g-2}(q-1)}{\chi(1)^{2g-2}} \frac{\chi(\xi_r^d)}{\chi(1)}$
- $\rightsquigarrow E(\mathcal{M}^{r;d}; q) = E(\mathcal{M}^{r;d'}; q)$ when $(d, r) = (d', r) = 1$
- $\mathcal{M}^{r;d}$ and $\mathcal{M}^{r;d'}$ Galois conjugate $\Rightarrow$ $H(\mathcal{M}^{r;d}; q, t) = H(\mathcal{M}^{r;d'}; q, t)$

Character table of $\mathrm{GL}_2(\mathbb{F}_q)$

Table 1: characters of $\mathbf{GL}_2(\mathbb{F}_q)$
(note that $|\mathbf{GL}_2(\mathbb{F}_q)| = q(q-1)^2(q+1)$)

Classes	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ $a \in \mathbb{F}_q^\times$	$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ $a, b \in \mathbb{F}_q^\times$, $a \neq b$	$\begin{pmatrix} x & 0 \\ 0 & {}^F x \end{pmatrix}$ $x \in \mathbb{F}_{q^2}^\times$, $x \neq {}^F x$	$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ $a \in \mathbb{F}_q^\times$
Number of classes of this type	$q-1$	$\frac{(q-1)(q-2)}{2}$	$\frac{q(q-1)}{2}$	$q-1$
Cardinal of the class	1	$q(q+1)$	$q(q-1)$	q^2-1
$R_{\mathbf{T}}^{\mathbf{G}}(\alpha,\beta)$, $\alpha,\beta \in \mathrm{Irr}(\mathbb{F}_q^\times)$, $\alpha \neq \beta$	$(q+1)\alpha(a)\beta(a)$	$\alpha(a)\beta(b) + \alpha(b)\beta(a)$	0	$\alpha(a)\beta(a)$
$-R_{\mathbf{T}_s}^{\mathbf{G}}(\omega)$, $\omega \in \mathrm{Irr}(\mathbb{F}_{q^2}^\times)$, $\omega \neq \omega^q$	$(q-1)\omega(a)$	0	$-\omega(x) - \omega({}^F x)$	$-\omega(a)$
$\mathrm{Id}_{\mathbf{G}}.(\alpha \circ \det)$, $\alpha \in \mathrm{Irr}(\mathbb{F}_q^\times)$	$\alpha(a^2)$	$\alpha(ab)$	$\alpha(x.{}^F x)$	$\alpha(a^2)$
$\mathrm{St}_{\mathbf{G}}.(\alpha \circ \det)$, $\alpha \in \mathrm{Irr}(\mathbb{F}_q^\times)$	$q\alpha(a^2)$	$\alpha(ab)$	$-\alpha(x.{}^F x)$	0

Example $\mathrm{GL}_2(\mathbb{F}_q)$

- $$\frac{1}{|\mathrm{PGL}_2(\mathbb{F}_q)|(q-1)^{2g}}\#\{a_j, b_j \in \mathrm{GL}_2(\mathbb{F}_q) | [a_1, b_1]\dots[a_g, b_g] = -1\} =$$

$$\frac{1}{|\mathrm{PGL}_2(\mathbb{F}_q)|(q-1)^{2g}} \sum_{\chi\in\mathrm{Irr}(\mathrm{GL}_2(\mathbb{F}_q))} \frac{|\mathrm{GL}_2(\mathbb{F}_q)|^{2g-1}}{\chi(1)^{2g-2}} \frac{\chi(-1)}{\chi(1)} =$$

$$(q^2-1)^{2g-2}+q^{2g-2}(q^2-1)^{2g-2}-\frac{1}{2}q^{2g-2}(q-1)^{2g-2}-\frac{1}{2}q^{2g-2}(q+1)^{2g-2}$$

$$\overset{Katz}{=} \frac{E(\mathcal{M}^2)}{(q-1)^{2g}}.$$

- e.g. $g = 0$ gives 0 when $g = 1$ it gives 1

Mixed Hodge polynomials of character varieties

Conjecture (Hausel-Villegas, 2008)

$$\sum_\lambda \prod \frac{\left(z^{2l+1}-w^{2a+1}\right)^{2g}}{(z^{2l+2}-w^{2a})(z^{2l}-w^{2a+2})} T^{|\lambda|} = \exp\left(\sum_{r,k} \frac{H\left(\mathcal{M}^r; w^{2k}, -(zw)^{-2k}\right)(zw)^{D_r}}{(z^{2k}-1)(1-w^{2k})} \frac{T^{rk}}{k}\right)$$

- when $g=0$ $\mathcal{M}^r$ empty unless $r=1$ when $\mathcal{M}^1 = pt \overset{HV}{\rightsquigarrow}$

$$\sum_\lambda \prod \frac{T^{|\lambda|}}{(z^{2l+2}-w^{2a})(z^{2l}-w^{2a+2})} = \exp\left(\sum_{k\geq 1} \frac{1}{(z^{2k}-1)(1-w^{2k})} \frac{T^k}{k}\right)$$

  proved by [Garsia–Haiman, 1996]

- when $g=1$ $\mathcal{M}^r = (\mathbb{C}^*)^2$ by Stone-von Neumann $\overset{HV}{\rightsquigarrow}$

$$\sum_\lambda \prod \frac{\left(z^{2l+1}-w^{2a+1}\right)^{2}}{(z^{2l+2}-w^{2a})(z^{2l}-w^{2a+2})} T^{|\lambda|} = \exp\left(\sum_{k\geq 1} \frac{(z^k-w^k)^2}{(z^{2k}-1)(1-w^{2k})(1-T^k)} \frac{T^k}{k}\right)$$

  proved geometrically by [Waelder, 2008]

Formula for $H(\mathcal{M}_{\mathrm{B}}^2; q, t)$

- when $r = 2$ Conjecture $\rightsquigarrow$

$$\frac{H_2(q,t)}{(1+qt)^{2g}} = \frac{(q^2t^3+1)^{2g}}{(q^2t^2-1)(q^2t^4-1)} + \frac{q^{2g-2}t^{4g-4}(q^2t+1)^{2g}}{(q^2-1)(q^2t^2-1)} - \\ - \frac{1}{2}\frac{q^{2g-2}t^{4g-4}(qt+1)^{2g}}{(qt^2-1)(q-1)} - \frac{1}{2}\frac{q^{2g-2}t^{4g-4}(qt-1)^{2g}}{(q+1)(qt^2+1)}$$

- $H(\mathcal{M}^2; q, t) \overset{HV}{=} H_2(q, t)$
- $H_2(q, -1) = H_2(1/q, -1)q^{8g-6} = E(\mathcal{M}^2; q)$
- $\frac{H_2(1,t)}{(1+t)^{2g}}$ agrees with [Hitchin 1987] for $P(\mathcal{M}_{\mathrm{B}}^2; t)$

$$P_t(\mathcal{M}) = \frac{(1+t^3)^{2g}}{(1-t^2)(1-t^4)} - \frac{t^{4g-4}(1-t)^{2g}}{4(1+t^2)} \\ - \frac{t^{4g-3}(1+t)^{2g-2}(g-1)}{(1-t)} + \frac{t^{4g-4}(1+t)^{2g-2}(t^2-4t+1)}{4(1-t)^2}$$

Consistency results

- [Hausel–Villegas 2008] $\rightsquigarrow$ for $r = 2$ $H(\mathcal{M}^2; q, t) = H_2(q, t)$ proved by finding the weights of universal generators and extending it to a monomial basis of [Hausel–Thaddeus 2002]
- [Hausel–Villegas 2008] $\rightsquigarrow$ for $r = 3$ conjecture is consistent with [Gothen, 1994] for $P(\mathcal{M}^3_{\mathrm{Dol}}; t)$
- [Garcia-Prada–Heinloth–Schmitt 2011] $\rightsquigarrow$ for $r = 4$ conjecture is consitent with their computation $P_t(\mathcal{M}^4_{\mathrm{Dol}})$
- [Garcia-Prada–Heinloth 2012] $\rightsquigarrow$ prove Hirzebruch y-genus specialization of extended conjecture for every r
- [Chuang–Diaconescu–Pan 2012] $\rightsquigarrow$ conjecture is equivalent with refined Gopakumar-Vafa conjecture for local curve Calabi-Yau 3-fold provided $P = W$

Character table for $\mathrm{GL}_3(\mathbb{F}_q)$

$|GL_3(\mathbb{F}_q)| = (q^3-1)(q^3-q)(q^3-q^2) = q^3(q-1)^3(q+1)(q^2+q+1)$

$|T^F| = (q-1)^3$, $|T_s^F| = (q^2-1)(q-1)$, $|T_\sigma^F| = q^3-1$

$|W(T)^F| = 6$, $|W(T_s)^F| = 2$, $|W(T_\sigma)^F| = 3$

Tables des caractères de $GL_3(\mathbb{F}_q)$

$|C_G\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}| = q^3(q-1)^2$

$|C_G\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}^F| = q^2(q-1)$

$|\mathcal{O}_G\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}| = (q^3-1)(q+1)$

$|\mathcal{O}_{G^F}\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}| = (q^3-1)(q^3-q)$

		$\begin{pmatrix}a&0&0\\0&a&0\\0&0&a\end{pmatrix}$ $a\in\mathbb{F}_q^\times$	$\begin{pmatrix}a&0&0\\0&a&0\\0&0&b\end{pmatrix}$ $a\neq b,\ a,b\in\mathbb{F}_q^\times$	$\begin{pmatrix}a&0&0\\0&b&0\\0&0&c\end{pmatrix}$ a,b,c 2 à 2 $\neq$ $\in\mathbb{F}_q^\times$	$\begin{pmatrix}t_1^q&0&0\\0&t_1&0\\0&0&t_2\end{pmatrix}$ $t_1\in\mathbb{F}_{q^2}^\times-\mathbb{F}_q$, $t_2\in\mathbb{F}_q^\times$	$\begin{pmatrix}t_1&0&0\\0&t_1^q&0\\0&0&t_1^{q^2}\end{pmatrix}$ $t_1\in\mathbb{F}_{q^3}^\times-\mathbb{F}_{q^2}$	$\begin{pmatrix}a&0&0\\0&a&1\\0&0&a\end{pmatrix}$ $a\in\mathbb{F}_q^\times$	$\begin{pmatrix}a&1&0\\0&a&1\\0&0&a\end{pmatrix}$ $a\in\mathbb{F}_q^\times$	$\begin{pmatrix}a&0&0\\0&b&1\\0&0&b\end{pmatrix}$ $a\neq b\in\mathbb{F}_q^\times$
	Nombre de classes de ce type	$q-1$	$(q-1)(q-2)$	$\frac{(q-1)(q-2)(q-3)}{6}$	$\frac{q(q-1)(q-1)}{6}$	$\frac{q(q^2-q)}{6}$	$q-1$	$q-1$	$(q-1)(q-2)$
	Cardinal des classes	1	$q^2(q^2+q+1)$	$q^3(q+1)(q^2+q+1)$	$q^2(q^2+q+1)(q^2-1)$	$(q^3-q)(q^3-q^2)$	$(q^3-1)(q+1)$	$(q^3-1)(q^3-q)$	$q^2(q^3-1)(q+1)$
6	$R_T^G(\alpha,\beta,\gamma)$ $\alpha,\beta,\gamma\in\mathrm{Irr}(\mathbb{F}_q^\times)$ (α,β,γ) 2 à 2 $\neq$	$(q+1)(q^2+q+1)\times\alpha(a)\beta(a)\gamma(a)$	$(q+1)(\alpha(a)\beta(a)\gamma(b)$ $+\alpha(b)\beta(a)\gamma(a)$ $+\alpha(a)\beta(b)\gamma(a))$	$\sum_{\sigma\in S_3}\alpha(\sigma a)\beta(\sigma b)\gamma(\sigma c)$	0	0	$(1+2q)\times\alpha(a)\beta(a)\gamma(a)$	$\alpha(a)\beta(a)\gamma(a)$	$\alpha(a)\beta(b)\gamma(b)$ $+\alpha(b)\beta(a)\gamma(b)$ $+\alpha(b)\beta(b)\gamma(a)$
1	$\mathrm{Id}_{G^F}.(\alpha\circ\det)$ $\alpha\in\mathrm{Irr}(\mathbb{F}_q^\times)$	$\alpha(a^3)$	$\alpha(a^2b)$	$\alpha(abc)$	$\alpha(t_1{}^Ft_1t_2)$	$\alpha(t_1{}^Ft_1{}^{F^2}t_1)$	$\alpha(a^3)$	$\alpha(a^3)$	$\alpha(ab^2)$
3	$\mathrm{St}_G.(\alpha\circ\det)$ $\alpha\in\mathrm{Irr}(\mathbb{F}_q^\times)$	$q^3\alpha(a^3)$	$q\alpha(a^2b)$	$\alpha(abc)$	$-\alpha(t_1{}^Ft_1t_2)$	$\alpha(t_1{}^Ft_1{}^{F^2}t_1)$	0	0	0
	$R_{T_\sigma}^G(w),\ w\neq w^q$ $w\in\mathrm{Irr}(\mathbb{F}_{q^3}^\times)$	$(q^2-1)(q-1)w(a)$	0	0	0	$w(t_1)+w(t_1^q)$ $+w(t_1^{q^2})$	$(1-q)w(a)$	$w(a)$	0
7	$-R_{T_s}^G(w,\alpha)$ $w\in\mathrm{Irr}(\mathbb{F}_{q^2}^\times),\neq w^q$ $\alpha\in\mathrm{Irr}(\mathbb{F}_q^\times)$	$(q^3-1)w(a)\alpha(a)$	$(q-1)w(a)\alpha(b)$	0	$-w(t_1)\alpha(t_2)$ $-w(t_1^q)\alpha(t_2)$	0	$-w(a)\alpha(a)$	$-w(a)\alpha(a)$	$-w(b)\alpha(b)$
2	$R_\theta.(\alpha\circ\det)$ $\alpha\in\mathrm{Irr}(\mathbb{F}_q^\times)$	$q(q+1)\alpha(a^3)$	$(q+1)\alpha(a^2b)$	$2\alpha(abc)$	0	$-\alpha(t_1{}^Ft_1{}^{F^2}t_1)$	$q\alpha(a^3)$	0	$\alpha(ab^2)$
4	$R_{C_G(s)}^G(\mathrm{Id}.(\alpha\circ\det,\beta))$ $\alpha\neq\beta\in\mathrm{Irr}(\mathbb{F}_q^\times)$	$(q^2+q+1)\alpha(a^2)\beta(a)$	$\alpha(a^2)\beta(b)+(q+1)\times\alpha(a)\alpha(b)\beta(a)$	$\alpha(ab)\beta(c)+[\alpha(ac)\beta(b)+\alpha(bc)\beta(a)]$	$\alpha(t_1^{q+1})\beta(t_2)$	0	$(q+1)\alpha(a^2)\beta(a)$	$\alpha(a^2)\beta(a)$	$\alpha(ab)\beta(b)$ $+\alpha(b^2)\beta(a)$
5	$R_{C_G(s)}^G(\mathrm{St}_{C_G(s)}.(\alpha\circ\det,\beta))$ $\alpha\neq\beta\in\mathrm{Irr}(\mathbb{F}_q^\times)$	$q(q^2+q+1)\times\alpha(a^2)\beta(a)$	$q\alpha(a^2)\beta(b)+(q+1)\times\alpha(a)\alpha(b)\beta(a)$	$\alpha(ab)\beta(c)$ $+\alpha(ac)\beta(b)$ $+\alpha(bc)\beta(a)$	$-\alpha(t_1^{q+1})\beta(t_2)$	0	$q.\alpha(a^2)\beta(a)$	0	$\alpha(b^2)\beta(a)$

Our conjecture for $H(\mathcal{M}^3; q, t)$

$$\frac{\left(q^3t^5-1\right)^{2g}\left(q^2t^3-1\right)^{2g}}{(q^3t^6-1)(q^3t^4-1)(q^2t^4-1)(q^2t^2-1)} + \frac{q^{6g-6}t^{12g-12}\left(q^3t-1\right)^{2g}\left(q^2t-1\right)^{2g}}{(q^3t^2-1)(q^3-1)(q^2t^2-1)(q^2-1)} +$$

$$+\frac{q^{4g-4}t^{8g-8}\left(q^3t^3-1\right)^{2g}(qt-1)^{2g}}{(q^3t^4-1)(q^3t^2-1)(qt^2-1)(q-1)} + \frac{1}{3}\,\frac{q^{6g-6}t^{12g-12}\left((qt-1)^{2g}\right)^2}{(qt^2-1)^2(q-1)^2} -$$

$$-\frac{1}{3}\,\frac{q^{6g-6}t^{12g-12}\left(q^2t^2+qt+1\right)^{2g}}{(q^2t^4+qt^2+1)(q^2+q+1)} - \frac{q^{4g-4}t^{8g-8}\left(q^2t^3-1\right)^{2g}(qt-1)^{2g}}{(q^2t^4-1)(q^2t^2-1)(qt^2-1)(q-1)} -$$

$$-\frac{q^{6g-6}t^{12g-12}\left(q^2t-1\right)^{2g}(qt-1)^{2g}}{(q^2t^2-1)(q^2-1)(qt^2-1)(q-1)}.$$

Gothen's formula for $P(\mathcal{M}^3_{\mathrm{Dol}}; t)$

$$
\begin{aligned}
P_t(\mathcal{M}) = & \frac{(1+t)^{4g-4}}{(1-t)^4}\Big(2t^2 + t^4 + 2t^{2g} + 2t^{2g+2} - \tfrac{1}{4}t^{4g-4} - 3gt^{4g-3} \\
& + (6g^2+2g-3)t^{4g-2} + (11g-12g^2)t^{4g-1} \\
& + (6g^2 - 10g + \tfrac{17}{4})t^{4g} - t^{8g-6} - t^{10g-8}\Big) \\
& + \frac{t^{2g}(1+t)^{2g-4}}{(1-t)^4(1+t^2)^2}\Big(t^{6g-8}(1+t^3)^{2g}(-2g - t^2 + (2g-2)t^4) \\
& + (1+t)^{2g}(-2t^4 - 2t^6 + t^{2g-4} + 2t^{2g-2} + t^{2g} - t^{4g-2})\Big) \\
& - \frac{2^{2g}t^{2g}(1+t)^{2g-1}}{(1-t)^4} + \frac{2gt^{8g-8}(1+t)^{2g-3}(1+t^3)^{2g-1}}{(1-t)^3(1+t^2)} \\
& + \frac{2^{2g-1}t^{10g-8}(1+t)^{2g}}{(1-t)^3(1-t^3)} + \frac{t^{4g-4}(1-t)^{2g-1}(1+t)^{2g-1}}{4(1+t^2)} \\
& + \frac{t^{6g-2}(1+t)^{4g-3}(1+t^2+t^4)}{(t-1)^3(1+t^2)^2(t^6-1)} + \frac{(1+t^5)^{2g}(1+t^3)^{2g-1}}{(t^2-1)(t^4-1)^2(t^3-1)}
\end{aligned}
$$

- the SL_n-character variety:

$$\mathcal{M}(\mathrm{SL}_n) := \{(A_i, B_i)_{i=1..g} \in \mathrm{SL}_n^{2g} \mid [A_1, B_1] \ldots [A_g, B_g] = \zeta_n^d I_n\} /\!/ \mathrm{PGL}_n$$

  non-singular, affine

- for PGL_n note that $\Gamma \cong (\mu_n)^{2g}$ acts on $\mathcal{M}(\mathrm{SL}_n)$

  $\mathcal{M}(\mathrm{PGL}_n) := \mathcal{M}(\mathrm{SL}_n)/\Gamma$

Theorem (Non-Abelian Hodge Theorem; Simpson, Corlette)

$$\mathcal{M}_{\mathrm{Dol}} \overset{diff}{\cong} \mathcal{M}$$

Conjecture (Hausel-Thaddeus 2003)

$$\begin{array}{ccc} H^*_{str}(\mathcal{M}_{\mathrm{Dol}}(\mathrm{SL}_n);\mathbb{Q}) & \cong & H^*_{str,B}(\mathcal{M}_{\mathrm{Dol}}(\mathrm{PGL}_n);\mathbb{Q}) \\ \cong & & \cong \\ \bigoplus_{\kappa\in\hat{\Gamma}} H^*(\mathcal{M}_{\mathrm{Dol}}(\mathrm{SL}_n);\mathbb{Q})_\kappa & \cong & \bigoplus_{\gamma\in\Gamma} H^{*-\mathrm{codim}\mathcal{M}^\gamma}(\mathcal{M}_{\mathrm{Dol}}(\mathrm{SL}_n)^\gamma/\Gamma, L^B_\gamma) \end{array}$$

Results:

- $n = 2, 3$ by (Hausel, Thaddeus 2003)
- for all n in the middle degree $* = \dim(\mathcal{M}_{\mathrm{Dol}}(\mathrm{SL}_n))$
  by (Garcia-Prada–Heinloth–Schmitt, 2010)
  using (Laumon, 1987)
- for all n up to degree $* < \min_{\gamma\in\Gamma^*} \mathrm{codim}\mathcal{M}^\gamma$
  by (Hausel–Pauly 2012)
  using symmetries of Hitchin fibers (Ngô, 2006)

Topological Betti Mirror Symmetry

Conjecture (Hausel-Villegas 2004)

$$\begin{array}{ccc} H^*_{str}(\mathcal{M}(\mathrm{SL}_n);\mathbb{Q}) & \cong & H^*_{str,B}(\mathcal{M}(\mathrm{PGL}_n);\mathbb{Q}) \\ \cong & & \cong \\ \bigoplus_{\kappa\in\hat{\Gamma}} H^*(\mathcal{M}(\mathrm{SL}_n);\mathbb{Q})_\kappa & \cong & \bigoplus_{\gamma\in\Gamma} H^{*-\mathrm{codim}\mathcal{M}^\gamma}(\mathcal{M}(\mathrm{SL}_n)^\gamma/\Gamma, L^B_\gamma) \end{array}$$

Conjecture (Refined mirror symmetry)

Under the Weil pairing $\Gamma^* \cong \Gamma$ *given by* $\kappa \leftrightarrow \gamma$

$$Gr^W_k H^*(\mathcal{M}(\mathrm{SL}_n))_\kappa \cong Gr^W_{2k-F(\gamma)} H^{*-F(\gamma)}(\mathcal{M}(\mathrm{SL}_n)^\gamma/\Gamma, L^B_\gamma)$$

in particular

$$\begin{array}{ccc} E_\kappa(\mathcal{M}(\mathrm{SL}_n)) & := & \sum_{i,k} |Gr^W_k H^i(\mathcal{M}(\mathrm{SL}_n))_\kappa| q^k (-1)^i \\ \| & & \| \\ E(\mathcal{M}(\mathrm{SL}_n)^\gamma/\Gamma, L^B_\gamma) q^{F(\gamma)} & = & \sum_{k,i} |Gr^W_{2k} H^i(\mathcal{M}(\mathrm{SL}_n)^\gamma/\Gamma, L^B_\gamma)| q^{k+F(\gamma)} (-1)^i \end{array}$$

Character formulas for refined Betti mirror symmetry

- $\kappa \in \hat{\Gamma}$ and $\epsilon \in \hat{\mathbb{F}}_q^\times$ such that $ord(\kappa) = ord(\epsilon) = k$

$$E_\kappa(\mathcal{M}(\mathrm{SL}_n)) = \sum_{\substack{\theta \in \mathrm{Irr}(\mathrm{SL}_n(\mathbb{F}_q)) \\ k \| \theta|}} |\theta|^{-2g} \left(\frac{|\mathrm{SL}_n(\mathbb{F}_q)|}{\theta(1)} \right)^{2g-2} \frac{\theta(\zeta_n 1)}{\theta(1)}$$

$$= \frac{1}{(q-1)^{2g-1}} \sum_{\substack{\chi \in \mathrm{Irr}(\mathrm{GL}_n(\mathbb{F}_q)) \\ \chi = \epsilon\chi}} \left(\frac{|\mathrm{GL}_n(\mathbb{F}_q)|}{\chi(1)} \right)^{2g-2} \frac{\chi(\zeta_n 1)}{\chi(1)}$$

- when $k = ord(\gamma)$

$$E(\mathcal{M}(\mathrm{SL}_n)^\gamma/\Gamma, L^B_\gamma) q^{F(\gamma)} =$$

$$\sum_{s|k} \frac{\mu(s) q^{n^2 \frac{k-1}{k}(g-1)}}{k(q-1)^{2g-1}} \sum_{\substack{\chi \in \mathrm{Irr}\left(\mathrm{GL}_{n/k}(\mathbb{F}_{q^s})\right) \\ \chi = \chi^{\mathrm{Frob}_q}}} \left(\frac{|\mathrm{GL}_{n/k}(\mathbb{F}_{q^s})|}{\chi(1)} \right)^{(2g-2)k/s} \left(\frac{\chi(\zeta_n 1)}{\chi(1)} \right)^{k/s}$$

Main result

Theorem (Hausel–Mereb–Villegas 2012)

When $\kappa \in \hat{\Gamma}$ and $\gamma \in \Gamma$ such that $ord(\kappa) = ord(\gamma) = k$

$$E(\mathcal{M}(\mathrm{SL}_n)^\gamma/\Gamma, L^B_\gamma)q^{F(\gamma)} =$$

$$= \frac{q^{n^2(g-1)}}{k(q-1)^{2g-2}} \sum_{u|k^\infty} \frac{1}{u} \sum_{s|k} \mu(s) \frac{q^{-\frac{(g-1)n^2}{uk}}}{(q^{us}-1)^2} E\left(\mathcal{M}^{(\frac{k}{s})}(\mathrm{GL}_{\frac{n}{ku}}); q^{us}\right) =$$

$$= E_\kappa(\mathcal{M}(\mathrm{SL}_n)).$$

- one can compute from this $\chi(\mathcal{M}(\mathrm{SL}_n)) = \mu(n)n^{4g-3}$ matching (Mereb 2010)
- this has a natural t-deformation giving a conjecture for the mixed Hodge polynomial of $\mathcal{M}(\mathrm{SL}_n)$ and in turn the mixed Hodge polynomial of $\mathcal{M}_{\mathrm{Dol}}(\mathrm{SL}_n)$

Example for Betti mirror symmetry for $n = 2$

- for $k = ord(\gamma) = 2$:

  $E(\mathcal{M}(\mathrm{SL}_2)^\gamma/\Gamma, L_\gamma^B)q^{F(\gamma)} =$

  $$\sum_{s|2} \frac{\mu(s)q^{2(g-1)}}{2(q-1)^{2g-1}} \sum_{\substack{\chi\in\mathrm{Irr}\left(\mathrm{GL}_1(\mathbb{F}_{q^s})\right)\\ \chi=\chi^{\mathrm{Frob}_q}}} \left(\frac{|\mathrm{GL}_1(\mathbb{F}_{q^s})|}{\chi(1)}\right)^{(2g-2)2/s} \left(\frac{\chi(\zeta_2 1)}{\chi(1)}\right)^{2/s}$$

  $$= \left(\frac{(q-1)^{2g-2}-(q+1)^{2g-2}}{2}\right) q^{2g-2}$$

- for $k = ord(\kappa) = 2$:

  $$E_\kappa(\mathcal{M}(\mathrm{SL}_2)) = \sum_{\substack{\theta\in\mathrm{Irr}(\mathrm{SL}_2(\mathbb{F}_q))\\ 2||\theta|}} |\theta|^{-2g} \left(\frac{|\mathrm{SL}_2(\mathbb{F}_q)|}{\theta(1)}\right)^{2g-2} \frac{\theta(\zeta_2 1)}{\theta(1)}$$

  $$= \left(\frac{(q-1)^{2g-2}-(q+1)^{2g-2}}{2}\right) q^{2g-2}$$

- character tables for $\mathrm{GL}_2(\mathbb{F}_q)$ and $\mathrm{SL}_2(\mathbb{F}_q)$ known to (Jordan 1907) and (Schur 1907)

Character table of $\mathrm{SL}_2(\mathbb{F}_q)$

Table 2: characters of $\mathbf{SL}_2(\mathbb{F}_q)$ for q odd
(note that $|\mathbf{SL}_2(\mathbb{F}_q)| = q(q-1)(q+1)$)

Classes	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ $a \in \{1,-1\}$	$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ $a \in \mathbb{F}_q^\times$ $a \neq \{1,-1\}$	$\begin{pmatrix} x & 0 \\ 0 & {}^F x \end{pmatrix}$ $x.{}^F x = 1$ $x \neq {}^F x$	$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ $a \in \{1,-1\}$, $b \in \{1, x\}$ with $x \in \mathbb{F}_q^\times - (\mathbb{F}_q^\times)^2$
Number of classes of this type	2	$(q-3)/2$	$(q-1)/2$	4
Cardinal of the class	1	$q(q+1)$	$q(q-1)$	$(q^2-1)/2$
$R_{\mathbf{T}}^{\mathbf{G}}(\alpha)$ $\alpha \in \mathrm{Irr}(\mathbb{F}_q^\times)$ $\alpha^2 \neq \mathrm{Id}$	$(q+1)\alpha(a)$	$\alpha(a) + \alpha(\frac{1}{a})$	0	$\alpha(a)$
$\chi_{\alpha_0}^{\varepsilon}$ $\varepsilon \in \{1,-1\}$	$\frac{q+1}{2}\alpha_0(a)$	$\alpha_0(a)$	0	$\frac{\alpha_0(a)}{2}(1- \varepsilon\alpha_0(ab)\sqrt{\alpha_0(-1)q})$
$-R_{\mathbf{T}_s}^{\mathbf{G}}(\omega)$ $\omega \in \mathrm{Irr}(\mu_{q+1})$ $\omega^2 \neq \mathrm{Id}$	$(q-1)\omega(a)$	0	$-\omega(x) - \omega({}^F x)$	$-\omega(a)$
$\chi_{\omega_0}^{\varepsilon}$ $\varepsilon \in \{1,-1\}$	$\frac{q-1}{2}\omega_0(a)$	0	$-\omega_0(x)$	$\frac{\omega_0(a)}{2}(-1+ \varepsilon\alpha_0(ab)\sqrt{\alpha_0(-1)q})$
$\mathrm{Id}_{\mathbf{G}}$	1	1	1	1
$\mathrm{St}_{\mathbf{G}}$	q	1	-1	0

Schur's character table of $SL_2(\mathbb{F}_q)$

2. Die Gruppe $\mathfrak{L}_{p^n}$, die durch die ganzen linearen Substitutionen

$$\xi_1 = \alpha\eta_1 + \beta\eta_2, \quad \xi_2 = \gamma\eta_1 + \delta\eta_2$$

gebildet wird, deren Determinante gleich 1 ist. — Die Ordnung der Gruppe

Die $s+4$ Charaktere von $\mathfrak{L}_s$ lassen sich in folgender Tabelle zusammenfassen:

	1	1	$\frac{s-3}{2}$	$\frac{s-1}{2}$	2	2
$\chi(E)$	1	s	$s+1$	$s-1$	$\frac{1}{2}(s+1)$	$\frac{1}{2}(s-1)$
$\chi(F)$	1	s	$(-1)^\alpha(s+1)$	$(-1)^\beta(s-1)$	$\frac{\varepsilon}{2}(s+1)$	$-\frac{\varepsilon}{2}(s-1)$
$\chi(P)$	1	0	1	-1	$\frac{1}{2}(1 \pm \sqrt{\varepsilon s})$	$\frac{1}{2}(-1 \pm \sqrt{\varepsilon s})$
$\chi(Q)$	1	0	1	-1	$\frac{1}{2}(1 \mp \sqrt{\varepsilon s})$	$\frac{1}{2}(-1 \mp \sqrt{\varepsilon s})$
$\chi(A^a)$	1	1	$\varrho^{\alpha a} + \varrho^{-\alpha a}$	0	$(-1)^a$	0
$\chi(B^b)$	1	-1	0	$-(\sigma^{\beta b} + \sigma^{-\beta b})$	0	$-(-1)^b$.

*) Vgl. *Dickson*, Linear Groups, Cap. XII.