

Symmetries of SL_n Hitchin fibers

joint work with Christian Pauly
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Main application of Harder-Narasimhan

- C , genus $g > 1$ smooth complex projective curve
- $\Lambda \in \text{Jac}^d(C)$ line bundle on C of degree d
- \mathcal{N} moduli space of semistable vector bundles on C of rank n and determinant Λ
- assume $(n, d) = 1 \Rightarrow \mathcal{N}$ smooth projective variety $\dim \mathcal{N} = (n^2 - 1)(g - 1)$
- $\Gamma := \text{Pic}(C)[n]$ order n line bundles on C
- $\Gamma \cong H^1(C, \mathbb{Z}(\text{SL}_n)) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$
- Γ acts on $\mathcal{N}: E \mapsto E \otimes L$

Theorem (Harder-Narasimhan 1975; Atiyah-Bott 1983)

The induced action of Γ on $H^(\mathcal{N}; \mathbb{Q})$ is trivial.*

$$H^*(\mathcal{N}; \mathbb{Q}) \cong H^*(\mathcal{N}; \mathbb{Q})^\Gamma \cong H^*(\mathcal{N}/\Gamma; \mathbb{Q})$$

- connected group H acts on topological space $Y \Rightarrow H^*(Y, \mathbb{Z})^H \cong H^*(Y, \mathbb{Z})$

SL_n & PGL_n Higgs moduli spaces

- C , genus $g > 1$ smooth complex projective curve
- $\Lambda \in \text{Jac}^d(C)$ line bundle on C of degree d
- Λ -twisted SL_n Higgs bundle (E, ϕ)
 $\phi \in H^0(C; \text{End}_0(E) \otimes K)$ and $\det(E) = \Lambda$
- \mathcal{M} moduli space of semi-stable Λ -twisted SL_n Higgs bundles
- $(n, d) = 1 \Rightarrow \mathcal{M}$ non-singular quasi-projective
 $\dim \mathcal{M} = 2(n^2 - 1)(g - 1)$
- $\Gamma := \text{Jac}(C)[n] \cong \mathbb{Z}_n^{2g}$ acts on \mathcal{M} : $(E, \phi) \mapsto (E \otimes L, \phi)$
- the PGL_n Higgs moduli space is the orbifold $\mathcal{M}_{PGL_n} = \mathcal{M}/\Gamma$
- (Hausel–Thaddeus 2001) $\sim \mathcal{M}_{SL_n} = \mathcal{M}$ and \mathcal{M}_{PGL_n} satisfy (Strominger-Yau-Zaslow 1996)

$$\begin{array}{ccc} \mathcal{M}_{SL_n} & & \mathcal{M}_{PGL_n} \\ & \searrow \chi_{SL_n} & \swarrow \chi_{PGL_n} \\ & \mathcal{A} & \end{array}$$

Conjecture (Hausel-Thaddeus 2001)

$$H^*(\mathcal{M}; \mathbb{Q}) \cong H_{st,B}^*(\mathcal{M}_{\mathrm{PGL}_n}; \mathbb{Q}) \cong \bigoplus_{\gamma \in \Gamma^*} H^{*-codim \mathcal{M}^\gamma}(\mathcal{M}^\gamma / \Gamma, L_\gamma^B) \oplus H^*(\mathcal{M} / \Gamma; \mathbb{Q})$$

- (Narasimhan-Ramanan 1974) $\Rightarrow (E, \phi) \in \mathcal{M}^\gamma \Leftrightarrow (E, \phi) = \pi_{\gamma*}(E', \phi')$ *endoscopic*
where (E', ϕ') stable Higgs bundle on $C_\gamma \xrightarrow{\pi_\gamma} C$

Corollary

$H^k(\mathcal{M}; \mathbb{Q}) \cong H^k(\mathcal{M} / \Gamma; \mathbb{Q}) \cong H^k(\mathcal{M}; \mathbb{Q})^\Gamma$
when $k < 2n^2(1 - 1/p)(g - 1) + 1$, $p|n$ smallest prime factor

- Hitchin map:
$$\begin{array}{ccc} \mathcal{M} & \rightarrow & \mathcal{A} := \bigoplus_{i=2}^n H^0(C; K^i) \\ (E, \phi) & \mapsto & \text{charpol}(\phi) \end{array}$$
- completely integrable system and proper
- $a = t^n + a_2 t^{n-2} + \dots + a_n \in \mathcal{A}$ where $a_i \in H^0(K^i)$
- X total space of K , $\pi : X \rightarrow C$
- $s_a = \lambda^n + a_2 \lambda^{n-2} + \dots + a_n \in H^0(X, \pi^* K^n)$
- $C_a := s_a^{-1}(0) \subset X$ spectral curve, $\pi_a : C_a \rightarrow C$ spectral cover ; could be smooth, integral, reducible, non-reduced
- (Simpson 1994) defines rank, degree and so a suitable stability condition for torsion-free coherent sheaves on C_a

Theorem (Beauville-Narasimhan-Ramanan 1989, Simpson 1994)

$\chi^{-1}(a) \cong$ the moduli space of rank 1 semi-stable torsion free sheaves \mathcal{E} on C_a , with $\det(\pi_*(\mathcal{E})) = \Lambda$

Prym varieties of spectral covers

- $\pi_a : C_a \rightarrow C$, norm map: $Nm_{C_a/C} : \text{Pic}^0(C_a) \rightarrow \text{Pic}^0(C)$
- main properties:

① group homomorphism; $Nm_{C_a/C}(\pi_a^*(L)) = L^n$

② $C_a = \cup X_i$, X_i irreducible $j_i = X_i^{red} \rightarrow X_i$ then

$$Nm_{C_a/C}(L) = \bigotimes Nm_{X_i^{red}/C}(j_i^*(L|_{X_i}))^{m_i}$$

③ \mathcal{E} rank 1 torsion-free, L invertible sheaf on C_a

$$\det((\pi_a)_*(\mathcal{E} \otimes L)) = \det((\pi_a)_*(\mathcal{E})) \otimes Nm_{C_a/C}(L)$$

- define $\text{Prym}(C_a/C) := Nm_{C_a/C}^{-1}(\mathcal{O}_C) \subset \text{Pic}^0(C_a)$
- $\text{Prym}(C_a/C)$ acts on $\chi^{-1}(a)$
- action of $\Gamma = \text{Pic}(C)[n]$ on $\chi^{-1}(a)$ factors through this action
- problem: $\pi_0(\text{Prym}(C_a/C)) = ?$
answered by (Ngô 2006) when C_a is integral

Main theorem

- C_a is a possibly reducible or non-reduced spectral curve
- $\gamma \in \Gamma^* \rightsquigarrow \pi_\gamma : C_\gamma \rightarrow C$ étale cover of order $o(\gamma) =: d|n$
- if (E, ϕ) stable rank n/d Higgs bundle on $C_\gamma \rightsquigarrow ((\pi_\gamma)_*E, (\pi_\gamma)_*\phi)$ is a stable rank n Higgs bundle on C called γ -endoscopic Higgs bundle
- $\mathcal{A}_\gamma \subset \mathcal{A}$ is the locus of $\chi((\pi_\gamma)_*\phi) \in \mathcal{A}$
- $K_a := \text{Hom}(\pi_0(\text{Prym}(C_a/C)) \rightarrow \mathbb{C}^\times) \subset \text{Pic}^0(C)[n]$

Theorem (Hausel-Pauly, 2011)

for all $a \in \mathcal{A}$ we have $\gamma \in K_a \Leftrightarrow a \in \mathcal{A}_\gamma$

Corollary

$\pi_0(\text{Prym}(C_a/C)) \neq 0 \Leftrightarrow a \in \mathcal{A}_{en} := \bigcup_{\gamma \in \Gamma^*} \mathcal{A}_\gamma$

Application to topological mirror symmetry

- if $o(\gamma) = d|n$ calculate $\dim(\mathcal{A}_\gamma) = (n^2/d - 1)(g - 1)$
- $\dim \mathcal{A} - \dim \mathcal{A}_{en} = n^2(1 - 1/p)(g - 1) := c$, where $p|n$ smallest prime
- $\mathcal{M}_{ne} := \chi^{-1}(\mathcal{A}_{ne})$, where $\mathcal{A}_{ne} := \mathcal{A} \setminus \mathcal{A}_{en}$
- (Hausel-Pauly 2011) $\Rightarrow \text{Prym}(C_a/C)$ is connected when $a \in \mathcal{A}_{ne} \Rightarrow \Gamma \subset \text{Prym}(C_a/C)$ acts trivially on $H^*(\chi^{-1}(a); \mathbb{Q})$
- $\Rightarrow \Gamma$ acts trivially on $H^*(\mathcal{M}_{ne}; \mathbb{Q})$
- $H^k(\mathcal{M}; \mathbb{Q}) \rightarrow H^k(\mathcal{M}_{ne}; \mathbb{Q})$ is injective if $k < 2c$

Corollary (Hausel-Pauly 2011)

Γ acts trivially on $H^k(\mathcal{M}; \mathbb{Q})$ when $k < 2c$

- (de Cataldo-Migliorini 2006) \Rightarrow

$$\ker(H^{2c}(\mathcal{M}; \mathbb{Q}) \rightarrow H^{2c}(\mathcal{M}_{ne}; \mathbb{Q})) \subset H_c^{2c}(\mathcal{M}; \mathbb{Q}) \stackrel{RHL}{\cong} H_{\dim \mathcal{M}-c}^{\dim \mathcal{M}}(\mathcal{M}; \mathbb{Q})$$

- (Garcia-Prada-Heinloth-Schmitt 2011) \Rightarrow
 Γ acts trivially on $H^{\dim \mathcal{M}}(\mathcal{M}; \mathbb{Q}) \Rightarrow$

Corollary (Hausel-Pauly 2011)

Γ acts trivially on $H^k(\mathcal{M}; \mathbb{Q})$ when $k < 2n^2(1 - 1/p)(g - 1) + 1$

- remains to extend Ngô's geometric fundamental lemma from \mathcal{A}_{int} to $\mathcal{A} \rightsquigarrow$ full topological mirror symmetry of (Hausel-Thaddeus 2001)

Application to Harder-Narasimhan

- \mathbb{C}^\times acts on $\mathcal{M}(E, \phi) \mapsto (E, \lambda\phi)$
- usual Morse theory: $H^*(\mathcal{M}; \mathbb{Q}) \cong \bigoplus_{F_i \subset \mathcal{M}^{\mathbb{C}^\times}} H^{*- \mu_i}(F_i; \mathbb{Q})$
- minimal stratum $F_0 = \mathcal{N} \subset \mathcal{M}$ with Morse index $\mu_0 = 0$
- $\Rightarrow \Gamma$ acts on $H^k(\mathcal{N}; \mathbb{Q})$ trivially when
 $k < 2n^2(1 - 1/p)(g - 1) + 1$
- $\dim \mathcal{N} = (n^2 - 1)(g - 1) < 2n^2(1 - 1/p)(g - 1)$
Poincaré duality \Rightarrow

Corollary (Harder-Narasimhan, 1975)

Γ acts trivially on $H^*(\mathcal{N}; \mathbb{Q})$.