Symmetries of SL_n Hitchin fibers

joint work with Christian Pauly
arXiv:1012.4748v2 and 1102.1717v2

Tamás Hausel

Royal Society URF at University of Oxford
http://www.maths.ox.ac.uk/~hausel/talks.html

Algebraic Geometry / Moduli Seminar
ETH, Zürich
9 December 2011

Main application of Harder-Narasimhan

- C, genus $g > 1$ smooth complex projective curve
- $\Lambda \in Jac^d(C)$ line bundle on C of degree d
- $\mathcal{N}$ moduli space of semistable vector bundles on C of rank n and determinant Λ
- assume $(n, d) = 1 \Rightarrow \mathcal{N}$ smooth projective variety $\dim \mathcal{N} = (n^2 - 1)(g - 1)$
- $\Gamma := \mathrm{Pic}(C)[n]$ order n line bundles on C
- $\Gamma \cong H^1(C, Z(\mathrm{SL}_n)) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$
- Γ acts on $\mathcal{N}$: $E \mapsto E \otimes L$

Theorem (Harder-Narasimhan 1975; Atiyah-Bott 1983)

The induced action of Γ on $H^*(\mathcal{N}; \mathbb{Q})$ is trivial.

$H^*(\mathcal{N}; \mathbb{Q}) \cong H^*(\mathcal{N}; \mathbb{Q})^\Gamma \cong H^*(\mathcal{N}/\Gamma; \mathbb{Q})$

- connected group H acts on topological space $Y \Rightarrow H^*(Y, \mathbb{Z})^H \cong H^*(Y, \mathbb{Z})$

Topological mirror symmetry

- Λ-twisted SL_n Higgs bundle (E, ϕ)
  $\phi \in H^0(C; \mathrm{End}_0(E) \otimes K)$ and $\det(E) = \Lambda$
- $\mathcal{M}$ moduli space of semi-stable Λ-twisted SL_n Higgs bundles
- $(n, d) = 1 \Rightarrow \mathcal{M}$ non-singular quasi-projective
  $\dim \mathcal{M} = 2(n^2 - 1)(g - 1)$
- Γ acts on $\mathcal{M}$: $(E, \phi) \mapsto (E \otimes L, \phi)$

Conjecture (Hausel-Thaddeus 2003)

$H^*(\mathcal{M}; \mathbb{Q}) \cong \bigoplus_{\gamma \in \Gamma^*} H^{*-\mathrm{codim}\mathcal{M}^\gamma}(\mathcal{M}^\gamma/\Gamma, L_\gamma^B) \oplus H^*(\mathcal{M}/\Gamma; \mathbb{Q})$

- (Narasimhan-Ramanan 1974) $\Rightarrow (E, \phi) \in \mathcal{M}^\gamma \Leftrightarrow$
  $(E, \phi) = \pi_{\gamma *}(E', \phi')$ *endoscopic*
  where (E', ϕ') stable Higgs bundle on $C_\gamma \stackrel{\pi_\gamma}{\to} C$

Corollary

$H^k(\mathcal{M}; \mathbb{Q}) \cong H^k(\mathcal{M}/\Gamma; \mathbb{Q})$
when $k < 2n^2(1 - 1/p)(g - 1) + 1$, $p|n$ *smallest prime factor*

Hitchin fibration

- Hitchin map: $\chi : \begin{array}{ccc} \mathcal{M} & \to & \mathcal{A} := \bigoplus_{i=2}^{n} H^0(C; K^i) \\ (E, \phi) & \mapsto & \mathrm{charpol}(\phi) \end{array}$
- completely integrable system and proper
- $a = t^n + a_2 t^{n-2} + \cdots + a_n \in \mathcal{A}$ where $a_i \in H^0(K^i)$
- X total space of K, $\pi : X \to C$
- $s_a = \lambda^n + a_2 \lambda^{n-2} + \cdots + a_n \in H^0(X, \pi^* K^n)$
- $C_a := s_a^{-1}(0) \subset X$ spectral curve, $\pi_a : C_a \to C$ spectral cover ; could be smooth, integral, reducible, non-reduced
- (Simpson 1994) defines rank, degree and so a suitable stability condition for torsion-free coherent sheaves on C_a

Theorem (Beauville-Narasimhan-Ramanan 1989, Simpson 1994)

$\chi^{-1}(a) \cong$ *the moduli space of rank* 1 *semi-stable torsion free sheaves* $\mathcal{E}$ *on* C_a*, with* $\det(\pi_*(\mathcal{E})) = \Lambda$

Prym varieties of spectral covers

- $\pi_a : C_a \to C$, norm map: $Nm_{C_a/C} : \mathrm{Pic}^0(C_a) \to \mathrm{Pic}^0(C)$
- main properties:
  1. group homomorphism; $Nm_{C_a/C}(\pi_a^*(L)) = L^n$
  2. $C_a = \cup X_i$, X_i irreducible $j_i = X_i^{red} \to X_i$ then
  $$Nm_{C_a/C}(L) = \bigotimes Nm_{X_i^{red}/C}(j_i^*(L|_{X_i}))^{m_i}$$
  3. $\mathcal{E}$ rank 1 torsion-free, L invertible sheaf on C_a
  $$\det((\pi_a)_*(\mathcal{E} \otimes L)) = \det((\pi_a)_*(\mathcal{E})) \otimes Nm_{C_a/C}(L)$$
- define $\mathrm{Prym}(C_a/C) := Nm^{-1}_{C_a/C}(\mathcal{O}_C) \subset \mathrm{Pic}^0(C_a)$
- $\mathrm{Prym}(C_a/C)$ acts on $\chi^{-1}(a)$
- action of $\Gamma = \mathrm{Pic}(C)[n]$ on $\chi^{-1}(a)$ factors through this action
- problem: $\pi_0(\mathrm{Prym}(C_a/C)) =?$
  answered by (Ngo 2006) when C_a is integral

Integral case

- $\pi_a : C_a \to C$ integral $\rightsquigarrow \tilde{\pi}_a : \tilde{C}_a \to C$ non-singular
- $\pi_0(\mathrm{Prym}(C_a/C)) \cong \pi_0(\mathrm{Prym}(\tilde{C}_a/C))$

$Nm_{\tilde{C}_a/C} : \mathrm{Pic}^0(\tilde{C}_a) \to \mathrm{Pic}^0(C)$ dual to $\tilde{\pi}_a^* : \mathrm{Pic}^0(\tilde{C}) \to \mathrm{Pic}^0(\tilde{C}_a)$

$\Rightarrow \pi_0(\mathrm{Prym}(\tilde{C}_a/C))$ dual to $\ker(\tilde{\pi}_a^*) \subset \mathrm{Pic}^0(C)[n]$

- $\pi_0(\mathrm{Prym}(C_a/C)) \neq 1 \rightsquigarrow 0 \neq \gamma \in \ker(\tilde{\pi}_a^*) \subset \mathrm{Pic}^0(C)[n]$

$\rightsquigarrow \tilde{C}_a \to C_\gamma \xrightarrow{\Gamma_\gamma} C$ factorizes

Integral case (contd)

- $\pi_0(\mathrm{Prym}(C_a/C)) \neq 1 \rightsquigarrow 0 \neq \gamma \in \ker(\tilde{\pi}_a^*) \subset \mathrm{Pic}^0(C)[n] \rightsquigarrow$
  $\tilde{C}_a \to C_\gamma \xrightarrow{\Gamma_\gamma} C$ factorizes

  $\rightsquigarrow C_{\gamma,a} := C_a \times_C C_\gamma = \bigcup_{\sigma \in \Gamma_\gamma} Z^\sigma \subset |K_{C_\gamma}|$ is Γ_γ-invariant spectral cover over C_γ, where $Z = im(\tilde{C}_a) \subset C_a \times_C C_\gamma$

  $\Rightarrow C_a = C_{\gamma,a}/\Gamma_\gamma$ is a γ-*endoscopic spectral curve* corresponding to the $\mathrm{GL}_{n/o(\gamma)}$ spectral curve $Z \subset |K_{C_\gamma}|$ on C_γ
- $\mathcal{A}_\gamma \subset \mathcal{A}$ locus of γ-endoscopic spectral curves

Theorem (Ngô, 2006)

when $a \in \mathcal{A}_{int}$

$$\gamma \in K_a \Leftrightarrow a \in \mathcal{A}_\gamma$$

where $K_a := \mathrm{Hom}(\pi_0(\mathrm{Prym}(C_a/C)) \to \mathbb{C}^\times) \subset \mathrm{Pic}^0(C)[n]$

General case

- C_a is a possibly reducible or non-reduced spectral curve
- one can reduce to the integral case using the identity

$$Nm_{C_a/C}(L) = \bigotimes Nm_{X_i^{red}/C}(j_i^*(L|_{X_i}))^{m_i}$$

where $C_a = \cup X_i$, X_i irreducible $j_i = X_i^{red} \to X_i$

Theorem (Hausel-Pauly, 2011)

for all $a \in \mathcal{A}$ we have $\gamma \in K_a \Leftrightarrow a \in \mathcal{A}_\gamma$

Corollary

$\pi_0(\mathrm{Prym}(C_a/C)) \neq 1 \Leftrightarrow a \in \mathcal{A}_{en} := \cup_{\gamma \in \Gamma^*} \mathcal{A}_\gamma$

Application to topological mirror symmetry

- if $o(\gamma) = d|n$ calculate $\dim(\mathcal{A}_\gamma) = (n^2/d - 1)(g - 1)$
- $\dim \mathcal{A} - \dim \mathcal{A}_{en} = n^2(1 - 1/p)(g - 1) := c$, where $p|n$ smallest prime
- $\mathcal{M}_{ne} := \chi^{-1}(\mathcal{A}_{ne})$, where $\mathcal{A}_{ne} := \mathcal{A} \setminus \mathcal{A}_{en}$
- (Hausel-Pauly 2011) $\Rightarrow \mathrm{Prym}(C_a/C)$ is connected when $a \in \mathcal{A}_{ne} \Rightarrow \Gamma \subset \mathrm{Prym}(C_a/C)$ acts trivially on $H^*(\chi^{-1}(a); \mathbb{Q})$
- $\Rightarrow \Gamma$ acts trivially on $H^*(\mathcal{M}_{ne}; \mathbb{Q})$
- $H^k(\mathcal{M}; \mathbb{Q}) \to H^k(\mathcal{M}_{ne}; \mathbb{Q})$ is injective if $k < 2c$

Corollary (Hausel-Pauly 2011)

Γ *acts trivially on* $H^k(\mathcal{M}; \mathbb{Q})$ *when* $k < 2c$

Application to topological mirror symmetry

- (de Cataldo-Migliorini 2006) $\Rightarrow$

$$\ker(H^{2c}(\mathcal{M};\mathbb{Q}) \to H^{2c}(\mathcal{M}_{ne};\mathbb{Q})) \subset H^{2c}_c(\mathcal{M};\mathbb{Q}) \overset{RHL}{\cong} H^{\dim\mathcal{M}}_{\dim\mathcal{M}-c}(\mathcal{M};\mathbb{Q})$$

- (Garcia-Prada-Heinloth-Schmitt 2011) $\Rightarrow$
  Γ acts trivially on $H^{\dim\mathcal{M}}(\mathcal{M};\mathbb{Q}) \Rightarrow$

Corollary (Hausel-Pauly 2011)

Γ acts trivially on $H^k(\mathcal{M};\mathbb{Q})$ when $k < 2n^2(1-1/p)(g-1)+1$

- remains to extend Ngô's geometric fundamental lemma from $\mathcal{A}_{int}$ to $\mathcal{A} \rightsquigarrow$ full topological mirror symmetry of (Hausel-Thaddeus 2003)

Application to Harder-Narasimhan

- $\mathbb{C}^\times$ acts on $\mathcal{M}$ $(E, \phi) \mapsto (E, \lambda\phi)$
- usual Morse theory: $H^*(\mathcal{M}; \mathbb{Q}) \cong \bigoplus_{F_i \subset \mathcal{M}^{\mathbb{C}^\times}} H^{*-\mu_i}(F_i; \mathbb{Q})$
- minimal stratum $F_0 = \mathcal{N} \subset \mathcal{M}$ with Morse index $\mu_0 = 0$
- $\Rightarrow \Gamma$ acts on $H^k(\mathcal{N}; \mathbb{Q})$ trivially when $k < 2n^2(1 - 1/p)(g - 1) + 1$
- $\dim \mathcal{N} = (n^2 - 1)(g - 1) < 2n^2(1 - 1/p)(g - 1)$
  Poincaré duality $\Rightarrow$

Corollary (Harder-Narasimhan, 1975)

Γ *acts trivially on* $H^*(\mathcal{N}; \mathbb{Q})$.