S-duality in hyperkähler Hodge theory

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Problem

Problem 1 (Hitchin, 1995)**.** *What is the space of L^2 harmonic forms on the moduli space of Higgs bundles on a Riemann surface?*

HyperKähLeR quotients

- Construction of (Hitchin-Karlhede-Lindström-Roček, 1987):
- $\mathbb{M}$ hyperkähler manifold
- G Lie group, $G \circlearrowright \mathbb{M}$ preserving the hyperkähler structure
- hyperkähler moment map:

$$\mu_{\mathbb{H}} = (\mu_I, \mu_J, \mu_K) : \mathbb{M} \to \mathbb{R}^3 \otimes \mathfrak{g}^*$$

- For $\xi \in \mathbb{R}^3 \otimes (\mathfrak{g}^*)^G$ the hyperkähler quotient

$$\mathbb{M}////_{\xi} G := \mu_{\mathbb{H}}^{-1}(\xi)/G,$$

has a natural hyperkähler metric at its smooth points

Moduli of Yang-Mills instantons on $\mathbb{R}^4$

- $P \to \mathbb{R}^4$ a $U(n)$-principal bundle over $\mathbb{R}^4$
- $\mathbb{M} =$
  $\{A \text{ connection on } P; |\int_{\mathbb{R}^4} tr(F_A \wedge *F_A)| < \infty\}$
- $A = A_1dx_1 + A_2dx_2 + A_3dx_3 + A_4dx_4$ in a fixed gauge, where $A_i \in V = \Omega^0(\mathbb{R}^4, \mathrm{adP})$
- $g \in \mathsf{G} = \Omega(\mathbb{R}^4, Ad(P))$ acts on $A \in \mathbb{M}$ by $g(A) = g^{-1}Ag + g^{-1}dg$, preserving the hyperkähler structure
- $\mu_{\mathbb{H}}(A) = 0 \Leftrightarrow F_A = *F_A$, self-dual Yang-Mills equation
- $\mathcal{M}(\mathbb{R}^4, P) = \mu_{\mathbb{H}}^{-1}(0)/\mathsf{G}$, the moduli space of finite energy self-dual Yang-Mills instantons on P, has a natural hyperkähler metric
- same story for X^4_{ALE} gravitational instanton $\Rightarrow \overline{\mathcal{M}}(X^4_{ALE}, P)$ Nakajima quiver variety

Moduli space of magnetic monopoles

- Assume that A_i are independent of x_4
- $A = A_1 dx_1 + A_2 dx_2 + A_3 dx_3$ connection on $\mathbb{R}^3$
- $A_4 = \phi \in \Omega^0(\mathbb{R}^3, \mathrm{ad}P)$ the *Higgs field*
- $\mathrm{G} = \Omega(\mathbb{R}^3, \mathrm{Ad}P) \circlearrowright \mathbb{M} = \{(A, \phi)$ + boundary cond.$\}$ preserving the natural hyperkähler metric on $\mathbb{M}$
- $\mu_{\mathbb{H}}(A, \phi) = 0 \Leftrightarrow F_A = *d_A\phi$ Bogomolny equation
- $\mathcal{M}(\mathbb{R}^3, P) = \mu_{\mathbb{H}}^{-1}(0)/\mathrm{G}$, the moduli space of magnetic monopoles on $\mathbb{R}^3$, has a natural hyperkähler metric
- Atiyah-Hitchin 1985 finds the metric explicitly on $\mathcal{M}^2(\mathbb{R}^3, P_{SU(2)}) \Rightarrow$ describe scattering of two monopoles

Moduli space of Higgs bundles

- Assume that A_i are independent of x_3, x_4
- $A = A_1 dx_1 + A_2 dx_2$ connection on $\mathbb{R}^2$
- $\Phi = (A_3 - A_4 i)dz \in \Omega^{1,0}(\mathbb{R}^2, \text{ad}P \otimes \mathbb{C})$ *complex Higgs field*
- $\mathsf{G} = \Omega(\mathbb{R}^2, \text{Ad}P) \circlearrowright \mathbb{M} = \{(A, \Phi)$ of finite energy $\}$ preserving the natural hyperkähler metric on $\mathbb{M}$
- the moment map equations

$$\mu_{\mathbb{H}}(A, \Phi) = 0 \Leftrightarrow \begin{array}{c} F(A) = -[\Phi, \Phi^*], \\ d''_A \Phi = 0. \end{array}$$

  equivalent with Hitchin's self-duality equations

- replacing $\mathbb{R}^2$ with a genus g compact Riemann surface C; $\mathcal{M}(C, P) = \mu_{\mathbb{H}}^{-1}(0)/\mathsf{G}$ has a natural hyperkähler metric

L^2 harmonic forms on complete manifolds

- M complete Riemannian manifold, $\alpha \in \Omega^k(M)$ is harmonic iff $d\alpha = d{*}\alpha = 0$; it is L^2 iff $\int_M \alpha \wedge {*}\alpha < \infty$; $\mathcal{H}^*(M)$ is the space of L^2 harmonic forms

- Hodge (orthogonal) decomposition:

$$\Omega^*_{L^2} = \overline{d(\Omega^*_{cpt})} \oplus \mathcal{H}^* \oplus \overline{\delta(\Omega^*_{cpt})},$$

- $H^*_{cpt}(M) \to \mathcal{H}^*(M) \to H^*(M)$ is the forgetful map

- Thus im$(H^*_{cpt}(M) \to H^*(M))$ "topological lower bound" for $\mathcal{H}^*(M)$

- $H^*_{cpt}(M) \to H^*(M)$ is equivalent with the intersection pairing on $H^*_{cpt}(M)$, by Poincaré duality

S-duality conjectures on L^2 harmonic forms

Conjecture 1 (Sen,1994).

$$SL(2,\mathbb{Z}) \circlearrowleft \bigcup_k \mathcal{H}^*(\widetilde{\mathcal{M}}_0^k(\mathbb{R}^3, P_{SU(2)}))$$

$$\Downarrow$$

$$\dim(\mathcal{H}^d(\widetilde{\mathcal{M}}_0^k(\mathbb{R}^3, P_{SU(2)}))) = \begin{cases} 0 & d \neq mid \\ \phi(k) & d = mid \end{cases}$$

Conjecture 2 (Vafa-Witten,1994). *Let* $M_\phi^{k,c_1} = \overline{\mathcal{M}}_\phi^{k,c_1}(X^4_{ALE}, P_{U(n)})$. *For* $d \neq mid$, $\dim(\mathcal{H}^d(M)) = 0$, *while*

$$\dim(\mathcal{H}^{mid}(M)) = \dim(\mathrm{im}(H^{mid}_{cpt}(M) \to H^{mid}(M))).$$

$$\Downarrow$$

$$Z_\phi(q) = \sum_{c_1,k} q^{k-c/24} \dim(\mathcal{H}^{mid}(M_\phi^{k,c_1}))$$ *is a modular form.*

Results on L^2 harmonic forms

- Sen 1994 $\Rightarrow$ L^2 harmonic 2-form on $\widetilde{\mathcal{M}_0^2}(\mathbb{R}^3, P_{SU(2)})$

- Segal-Selby 1996 $\Rightarrow$ dim(im($H^{mid}_{cpt}(M) \stackrel{\cong}{\to} H^{mid}(M))) = \phi(k)$ for $M = \widetilde{\mathcal{M}_0^k}(\mathbb{R}^3, P_{SU(2)})$

- Hausel 1998 $\Rightarrow$ dim(im($H^{mid}_{cpt}(M) \to H^{mid}(M))) = 0$ for $g > 1$ and $M = \mathcal{M}^1_{\mathrm{Dol}}(SL(2,\mathbb{C}))$

- Hitchin 2000 $\Rightarrow$ $\mathcal{H}^d(M) = 0$ unless $d = mid$; for a complete hyperkähler manifold of linear growth; proves Sen's conjecture for $k = 2$

- Hausel-Hunsicker-Mazzeo 2002 proves for fibered boundary manifolds M (like ALE, ALF and some ALG gravitational instantons) $\mathcal{H}^{mid}(M) = \mathrm{im}(IH^{mid}_{\underline{m}}(\overline{M}) \to IH^{mid}_{\overline{m}}(\overline{M}))$

- Carron 2005 proves for a QALE space M: $\mathcal{H}^{mid}(M) = \mathrm{im}(H^{mid}_{cpt}(M) \to H^{mid}(M))$

Mixed Hodge Structure of Deligne

- $\bigoplus_{p,q} H^{p,q;k}(M)$ is the associated graded to the weight and Hodge filtrations on the cohomology $H^k(M, \mathbb{C})$ of a complex algebraic variety M

- $h^{p,q;k} = \dim(H^{p,q;k}(M))$, the *mixed Hodge numbers*

- $H(M; x, y, t) = \sum_{p,q,k} h^{p,q;k}(M) x^p y^q t^k$, the *mixed Hodge polynomial*

- $P(M; t) = H(M; 1, 1, t)$, the *Poincaré polynomial*

- $E(M; x, y) = x^n y^n H(1/x, 1/y, -1)$, the *E-polynomial* of a smooth variety M.

Arithmetic and topological content of the E-polynomial

Theorem 2 (Katz 2005). *If M is a smooth quasi-projective variety defined over $\mathbb{Z}$ and*

$$\#\{M(\mathbb{F}_q)\} = E(q)$$

is a polynomial in q, then

$$E(M; x, y) = E(xy).$$

- MHS on $H^*(M,\mathbb{C})$ is *pure* if $h^{p,q;k} = 0$ unless $p+q = k \Leftrightarrow H(M;x,y,t) = (xyt^2)^n E(\frac{-1}{xt}, \frac{-1}{yt}) \Rightarrow P(M;t) = H(M;1,1,t) = t^{2n} E(\frac{-1}{t}, \frac{-1}{t})$; examples of varieties with pure MHS: smooth projective varieties, $\mathcal{M}_{\text{Dol}}$, $\mathcal{M}_{\text{DR}}$, Nakajima's quiver varieties

- the pure part of $H(M;x,y,t)$ is $PH(M;x,y) = \text{Coeff}_{T^0}\left(H(M;xT,yT,tT^{-1}\right)$ for a smooth M, it is always the image of the cohomology of a smooth compactification

Nakajima quiver varieties

- Γ quiver with vertex set I and edges $E \subset I \times I$; $\mathbf{v}, \mathbf{w} \in \mathbb{N}^I$ two dimension vectors; V_i and W_i corresponding vector spaces

- $\mathbb{V}_{\mathbf{v},\mathbf{w}} = \bigoplus_{a \in E} \mathsf{Hom}(V_{t(a)}, V_{h(a)}) \oplus \bigoplus_{i \in I} \mathsf{Hom}(V_i, W_i)$, action $\mathsf{GL}(\mathbf{v}) = \prod_{i \in I} \mathsf{GL}(V_i) \to \mathsf{GL}(\mathbb{V}_{\mathbf{v}})$

- for $\xi = 1_{\mathbf{v}} \in \mathfrak{gl}(\mathbf{v})^{\mathsf{GL}(\mathbf{v})}$ define the (always smooth) Nakajima quiver variety by

$$\mathcal{M}(\mathbf{v}, \mathbf{w}) = \mathbb{V}_{\mathbf{v},\mathbf{w}} \times \mathbb{V}^*_{\mathbf{v},\mathbf{w}} ////_{\xi} \mathsf{GL}(\mathbf{v})$$

Theorem 3 (Nakajima 1998)**.** *There is an irreducible representation of the Kac-Moody algebra $\mathfrak{g}(\Gamma)$ of highest weight $\mathbf{w}$ on $\oplus_{\mathbf{v}} H^{mid}(\mathcal{M}(\mathbf{v}, \mathbf{w}))$. In particular the Weyl-Kac character formula gives the middle Betti numbers of Nakajima quiver varieties. When Γ affine Dynkin diagram $\mathcal{M}(\mathbf{v}, \mathbf{w})$ is a component of $\overline{\mathcal{M}}(X_{\Gamma}, U(n)$. In the affine case the Weyl-Kac character formula is known to have modular properties $\Rightarrow$ Vafa-Witten.*

Theorem 4 (Hausel 2005). *For any quiver Γ, the Betti numbers of the Nakajima quiver varieties are:*

$$\sum_{\mathbf{v}\in\mathbb{N}^I} P_t(\mathcal{M}(\mathbf{v},\mathbf{w}))t^{-d(\mathbf{v},\mathbf{w})}T^{\mathbf{v}} = \frac{\displaystyle\sum_{\mathbf{v}\in\mathbb{N}^I} T^{\mathbf{v}} \sum_{\lambda\in\mathcal{P}(\mathbf{v})} \frac{\left(\prod_{(i,j)\in E} t^{-2\langle\lambda^i,\lambda^j\rangle}\right)\left(\prod_{i\in I} t^{-2\langle\lambda^i,(1^{\mathbf{w}_i})\rangle}\right)}{\prod_{i\in I}\left(t^{-2\langle\lambda^i,\lambda^i\rangle}\prod_k\prod_{j=1}^{m_k(\lambda^i)}(1-t^{2j})\right)}}{\displaystyle\sum_{\mathbf{v}\in\mathbb{N}^I} T^{\mathbf{v}} \sum_{\lambda\in\mathcal{P}(\mathbf{v})} \frac{\prod_{(i,j)\in E} t^{-2\langle\lambda^i,\lambda^j\rangle}}{\prod_{i\in I}\left(t^{-2\langle\lambda^i,\lambda^i\rangle}\prod_k\prod_{j=1}^{m_k(\lambda^i)}(1-t^{2j})\right)}},$$

Corollary 5. *The RHS is a deformation of the Weyl-Kac character formula* $\Rightarrow A_\Gamma(\mathbf{v},0)=m_{\mathbf{v}}$ *proving Kac's conjecture (1982) , where*

$$A_\Gamma(\mathbf{v},q) := \left|\left\{\begin{array}{c}\text{abs. indec. reps of } \Gamma \text{ over } \mathbb{F}_q \\ \text{of dimension } \mathbf{v}, \text{ modulo isomorphism}\end{array}\right\}\right|$$

Corollary 6. *When the quiver is affine ADE the RHS becomes an infinite product* $\Rightarrow$ *"elementary" proof of the modularity in the Vafa-Witten S-duality conjecture*

Spaces diffeomorphic to $\mathcal{M}(C, P_{U(n)})$

$$\mathcal{M}^d_{\mathsf{Dol}}(GL(n,\mathbb{C})) := \left\{ \begin{array}{c} \text{moduli space of semistable rank } n \\ \text{degree } d \text{ Hitchin pairs on } C \end{array} \right\}$$

$$\mathcal{M}^d_{\mathsf{DR}}(GL(n,\mathbb{C})) := \left\{ \begin{array}{c} \text{moduli space of flat } GL(n,\mathbb{C})\text{-connections} \\ \text{on } C \setminus \{p\}, \text{ with holonomy } e^{\frac{2\pi i d}{n}} Id \text{ around } p \end{array} \right\}$$

$$\mathcal{M}^d_{\mathsf{B}}(GL(n,\mathbb{C})) := \{A_1, B_1, \dots, A_g, B_g \in GL(n,\mathbb{C}) | \\ A_1^{-1}B_1^{-1}A_1B_1 \dots A_g^{-1}B_g^{-1}A_gB_g = \xi_n Id\}/GL(n,\mathbb{C})$$

Topological Mirror Test

Theorem 7 (Hausel–Thaddeus 2002). *In the following diagram*

$$\begin{array}{ccc} \mathcal{M}^d_{\mathsf{Dol}}(PGL(n)) & \longrightarrow & \mathcal{M}^d_{\mathsf{Dol}}(SL(n)) \\ \downarrow\chi_{PGL(n)} & & \downarrow\chi_{SL(n)} \\ \mathcal{H}_{PGL(n)} & \cong & \mathcal{H}_{SL(n)}. \end{array}$$

the generic fibers of the Hitchin maps $\chi_{PGL(n)}$ and $\chi_{SL(n)}$ are dual Abelian varieties. $\Rightarrow$ $\mathcal{M}^d_{\mathsf{DR}}(PGL(n))$ and $\mathcal{M}^d_{\mathsf{DR}}(SL(n))$ satisfy the SYZ construction for a pair of mirror symmetric Calabi-Yau manifolds.

Conjecture 3 (Hausel–Thaddeus 2002). *For all $d, e \in \mathbb{Z}$, satisfying $(d, n) = (e, n) = 1$,*

$$E^{B^e}_{\mathsf{st}}\left(x, y; \mathcal{M}^d_{\mathsf{DR}}(SL(n, \mathbb{C}))\right) = E^{\hat{B}^d}_{\mathsf{st}}\left(x, y; \mathcal{M}^e_{\mathsf{DR}}(PGL(n, \mathbb{C}))\right),$$

which morally should be related to S-duality in the recent work (Kapustin-Witten 2006) about a physical interpration of the Geometric Langlands programme.

Mirror symmetry for finite groups of Lie type

Conjecture 4 (Hausel–R-Villegas 2004).

$$E_{\mathsf{st}}^{B^e}\Big(x, y, \mathcal{M}_{\mathsf{B}}^d(SL(n,\mathbb{C}))\Big) = E_{\mathsf{st}}^{\hat{B}^d}\left(x, y, \mathcal{M}_{\mathsf{B}}^e(PGL(n,\mathbb{C}))\right)$$

Theorem 8 (Hausel–R-Villegas, 2004). *$G = SL(n)$ or $GL(n)$ $G(\mathbb{F}_q)$ finite group of Lie type*

$$E(\sqrt{q}, \sqrt{q}, \mathcal{M}_B^d(G)) = \#\{\mathcal{M}_B^d(G(\mathbb{F}_q))\} = \sum_{\chi \in Irr(G(\mathbb{F}_q))} \frac{|G(\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-1}} \chi(\xi_n^d)$$

$$\Downarrow$$

" *differences between the character tables of $PGL(n,\mathbb{F}_q)$ and its Langlands dual $SL(n,\mathbb{F}_q)$ are governed by mirror symmetry*"

It follows from (Hausel–Thaddeus 2000):

$$H(\mathcal{M}_B(PGL(2,\mathbb{C}));\sqrt{q},\sqrt{q},t) = \frac{(q^2t^3+1)^{2g}}{(q^2t^2-1)(q^2t^4-1)}+$$
$$+\frac{q^{2g-2}t^{4g-4}(q^2t+1)^{2g}}{(q^2-1)(q^2t^2-1)}-\frac{1}{2}\frac{q^{2g-2}t^{4g-4}(qt+1)^{2g}}{(qt^2-1)(q-1)}-\frac{1}{2}\frac{q^{2g-2}t^{4g-4}(qt-1)^{2g}}{(q+1)(qt^2+1)},$$

when $g=3$ this equals:

$$t^{12}q^{12}+t^{12}q^{10}+6\,t^{11}q^{10}+t^{12}q^8+t^{10}q^{10}+6\,t^{11}q^8+16\,t^{10}q^8+6\,t^9q^8+t^{10}q^6+$$
$$+t^8q^8+26\,t^9q^6+16\,t^8q^6+6\,t^7q^6+t^8q^4+t^6q^6+6\,t^7q^4+16\,t^6q^4+$$
$$+6\,t^5q^4+t^4q^4+t^4q^2+6\,t^3q^2+t^2q^2+1.$$

Corollary 9 (Hausel, 2005 & 2000 ⇒1998). *Newstead's $\beta^g=0 \Rightarrow PH^{mid}_{cpt}$ is trivial ⇒ trivial intersection form on $H^*_{cpt}(\mathcal{M}^1_B(PGL(2,\mathbb{C})))$.*

Answer

Conjecture 5 ("Purity conjecture" Hausel 2006)**.** *Studying the Riemann-Hilbert monodromy map* $\mathcal{M}_{DR} \stackrel{RH}{\to} \mathcal{M}_B$ *on the level of mixed Hodge structures in the parabolic case* $\Rightarrow$

$$q^{-mid} PP(\mathcal{M}_{\mathsf{B}}, q^{-1/2}) = A_{\Gamma}(\mathbf{v}, q),$$

where $(\Gamma, \mathbf{v})$ *is the star-shaped quiver and dimension vector given by the parabolic structure.*

Corollary 10. *Let* $M = \mathcal{M}_{Dol}$ *moduli of stable parabolic Higgs bundles, and* $\chi_{L^2}(M) = \dim(\mathrm{im}(H^{mid}_{cpt}(M) \to H^{mid}(M)))$ *then*

$$\chi_{L^2}(M) = 0 \text{ when } g > 1$$

$$\chi_{L^2}(M) = 1 \text{ when } g = 1$$

$$\chi_{L^2}(M) = A_{\Gamma}(\mathbf{v}, 0) = m_{\mathbf{v}}, \text{ when } g = 0$$

which are encoded by the Kac dominator formula for the star-shaped quiver Γ*.*