

# Refined geometric invariants and representation theory

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# Recollection on Frobenius algebras

- $\mathbb{K}$  field of characteristic 0
- *Frobenius algebra*: finite dimensional commutative unital  $\mathbb{K}$ -algebra  $F$  with non-degenerate pairing  $\langle, \rangle$ , which is symmetric  $\langle a, bc \rangle = \langle ab, c \rangle$
- e.g.  $\mathbb{K}R(G)$  for finite group  $G$  or  $\mathbb{K}[G]^G$  with convolution
- 1+1D TQFT  $\Leftrightarrow Z(S^1) = F$  with pairing  $\langle, \rangle$
- $(a_i)$  basis of orthogonal idempotents then

$$Z(\Sigma_g) = \sum_i \langle a_i, 1 \rangle^{1-g} \text{ Verlinde formula}$$

- e.g. when  $F = \mathbb{K}[G]^G$  we get  $(\chi |G|^{-\frac{1}{2}})_{\chi \in \hat{G}}$  orthogonal idempotents and

$$Z(\Sigma_g) = \sum_{\chi \in \hat{G}} \left( \frac{|G|^2}{\chi(1)^2} \right)^{g-1} = \frac{1}{|G|} |\text{Hom}(\pi_1(\Sigma_g), G)|$$

- 1+1D Chern-Simons theory with finite gauge group of (Freed–Quinn, 1993)

- $C$  genus  $g$  curve; fix group  $GL_n$

$$\mathcal{M}_{\text{Dol}}^d := \left\{ \begin{array}{l} \text{moduli space of semistable rank } n \\ \text{degree } d \text{ Higgs bundles } (E, \phi) \\ \phi \in H^0(C, \text{End}(E) \otimes K) \text{ Higgs field} \end{array} \right\}$$

$$\mathcal{M}_{\text{B}}^d := \{A_1, B_1, \dots, A_g, B_g \in GL_n \mid \prod_{i=1}^g A_i^{-1} B_i^{-1} A_i B_i = e^{\frac{2\pi i d}{n}} Id\} // PGL_n$$

when  $(d, n) = 1$  these are smooth non-compact varieties

- Non-Abelian Hodge Theorem:  $\mathcal{M}_{\text{Dol}}^d \stackrel{\text{diff}}{\cong} \mathcal{M}_{\text{B}}^d$   
(Hitchin, Donaldson, Corlette, Simpson)

- (Deligne 1971) defines weight filtration  $W_0 \subset \cdots \subset W_i \subset \cdots \subset W_{2k} = H_c^k(X; \mathbb{Q})$  for any complex algebraic variety  $X$
- $WH(X; q, t) = \sum \dim(W_i/W_{i-1}(H_c^k(X))) t^k q^{\frac{i}{2}}$ , mixed Hodge polynomial
- $P_c(X; t) = WH(X; 1, t)$ , Poincaré polynomial
- $WH(X; q, -1)$ , virtual weight polynomial of  $X$

## Theorem (Katz 2008)

If  $M$  is a smooth quasi-projective variety defined over  $\mathbb{Z}$  and

$$\#\{M(\mathbb{F}_q)\} = E(q)$$

is a polynomial in  $q$ , then  $WH(M(\mathbb{C}); q, -1) = E(q)$ .

# Conjectures on $WH(\mathcal{M}_B^d; q, t)$

- (Hausel-Villegas 2008) calculates  $WH(\mathcal{M}_B^d; q, -1) = |\mathcal{M}_B^d(\mathbb{F}_q)| = \sum_{\chi \in Irr(\mathrm{GL}_n(\mathbb{F}_q))} \frac{|\mathrm{GL}_n(\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-2}} \frac{\chi(\xi_n^d)}{\chi(1)}$
- thus  $WH(\mathcal{M}_B^d; q, -1)$  computable from Frobenius algebra  $\mathbb{C}[\mathrm{GL}_n(\mathbb{F}_q)]^{\mathrm{GL}_n(\mathbb{F}_q)}$

## Conjecture (Hausel-Villegas, 2008)

$$\sum_{\lambda} \prod \frac{(z^{2l+1} - w^{2a+1})^{2g}}{(z^{2l+2} - w^{2a})(z^{2l} - w^{2a+2})} T^{|\lambda|} = \exp \left( \sum_{n,k} \frac{WH(\mathcal{M}_B^{n,d}; w^{2k}, -(zw)^{-2k})}{(z^{2k}-1)(1-w^{2k})(zw)^{-dn}} \frac{T^{nk}}{k} \right)$$

- checks:
  - 1  $z = 1/w$  and  $n = 2$  (Hausel-Villegas, 2008)
  - 2  $w = 1$  (Chuang-Diaconescu-Pan 2010) from string theory arguments via (Mozgovoy 2011)
  - 3  $w = 1$ ? (Schiffmann 2015) computes  $P_c(\mathcal{M}_{\mathrm{Dol}}^d; t)$
  - 4  $w = 1$  (Maulik-Pixton 2016) rigorizing (CDP 2010)
- Is there a Frobenius algebra and so representation theory behind  $WH(\mathcal{M}_B^d; q, t)$ ?

# Conjectures on $PH(\mathcal{M}_{\text{Dol}}^d; q, t)$

- (De Cataldo–Migliorini 2005)  $\rightsquigarrow$   
 $P_0 \subset \cdots \subset P_k = H_c^k(\mathcal{M}_{\text{Dol}}^d; \mathbb{Q})$  perverse filtration induced by the Hitchin map  $\chi : \mathcal{M}_{\text{Dol}}^d \rightarrow \mathcal{A}$
- $P(\mathcal{M}_{\text{Dol}}^d; q, t) := \sum \dim(P_i/P_{i-1}(H_c^k(X))) t^k q^i$   
perverse Hodge polynomial

## Conjecture (De Cataldo–Hausel–Migliorini 2012)

$P_i(H_c^*(\mathcal{M}_{\text{Dol}}^d)) = W_{2i}(H_c^*(\mathcal{M}_{\text{B}}^d))$  in particular  
 $PH(\mathcal{M}_{\text{Dol}}^d; q, t) = WH(\mathcal{M}_{\text{B}}^d; q, t)$

- checks:
  - ①  $n = 2$  (De Cataldo–Hausel–Migliorini 2012)
  - ② (Chuang–Diaconescu–Pan 2013) string theoretical argument checking conjecture for  $PH(\mathcal{M}_{\text{Dol}}^d; q, t)$
- Is there a Frobenius algebra and so representation theory computing  $PH(\mathcal{M}_{\text{Dol}}^d; q, t)$ ?

- $C$  smooth complex projective curve of genus  $g$
- fix rank  $n \in \mathbb{Z}_{>0}$ , degree  $d \in \mathbb{Z}$  and level  $k \in \mathbb{Z}_{>0}$
- $\check{\mathcal{N}}_n^d$  moduli space of semi-stable rank  $n$  fixed degree  $d$  vector bundles on  $C$ ; projective and smooth when  $(d, n) = 1$
- $L \in \text{Pic}(\check{\mathcal{N}}_n^d) \cong \mathbb{Z}$  ample generator of Picard group
- Verlinde formula (1988) for  $\dim H^0(\check{\mathcal{N}}_n^d; L^k) = \chi(\check{\mathcal{N}}_n^d, L^k)$
- e.g. for  $n = 2$   $d = 1$

$$\dim H^0(\check{\mathcal{N}}_2^1, L^k) = \sum_{j=1}^{2k+1} (-1)^{j+1} \left( \frac{k+1}{\sin^2(\frac{j\pi}{2k+2})} \right)^{g-1} =$$

$$\frac{1}{2} \text{Res}_{z=1} \frac{(4k+4)^g}{(z^{k+1} - z^{-(k+1)})((1-1/z)(1-z))^{g-1}} \frac{dz}{z}$$

- proved for
  - $k=1$  by (Beauville–Narasimhan–Ramanan 1988)
  - $n=2$  by (Szenes, Bertram–Szenes 1993)
  - $\vdots$
  - in all generality by (Teleman–Woodward, 2009)

- Verlinde formula = partition function of a  $1 + 1D$  TQFT
- $R(\mathrm{SU}_n) \cong$  character ring of  $\mathrm{SU}_n$
- $R(\mathrm{SU}_n) \cong R(T_n)^{S_n} \cong (\mathbb{Z}[z_1, \dots, z_n]/(z_1 \cdots z_n - 1))^{S_n}$   
 irrep  $\chi_\lambda \in \mathrm{Irr}(\mathrm{SU}_n) \xrightarrow{\text{BWB}} \chi(\mathcal{F}; L_\lambda) = s_\lambda \in R(T_n)^{S_n}$  Schur fn  
 $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq 0) \in \mathbb{Z}^{n-1}$
- $\mathrm{Ver}_n^k := \mathbb{C}R/(s_{(k+1)}, s_{(k+2)}, \dots, s_{(k+n-1)})$  has basis  $\{s_\lambda\}_{\lambda_1 \leq k}$
- declaring  $\langle s_\lambda, s_{\eta^\dagger} \rangle = \delta_{\lambda\eta} \rightsquigarrow$  non-degenerate pairing

Theorem (Goodman-Wenzl 1990; Gepner 1991; Witten 1993)

$(\mathrm{Ver}_n^k, \langle, \rangle)$  is a Frobenius algebra (i.e.  $\langle a, bc \rangle = \langle ab, c \rangle$ ).  
 $\cong$  Verlinde algebra, with partition function giving Verlinde formulae

- the Frobenius algebra  $(\mathrm{Ver}_n^k, \langle, \rangle)$  arises as
  - 1 fusion algebra of level  $k$  representations of  $\widehat{\mathfrak{sl}}_n$
  - 2 representation ring of  $U_{e^{2\pi i(k+n)}}(\mathfrak{sl}_n)$
  - 3 representation ring of  $H_n(e^{2\pi i(k+n)})$



- $\check{\mathcal{M}}_n^d \supset T^*\check{\mathcal{N}}_n^d$  moduli ss rank  $n$  fixed degree  $d$  Higgs bundles
- $\mathbb{T} := \mathbb{C}^\times$  acts on  $\check{\mathcal{M}}_n^d$  by scaling Higgs field
- $L \in \text{Pic}(\check{\mathcal{M}}_n^d)$  ample generator with  $\mathbb{T}$  action trivial on  $L^k|_{\check{\mathcal{N}}_n^d}$
- $\mathbb{T}$  acts on  $H^0(\check{\mathcal{M}}_n^d, L^k)$  with weights  $\geq 0$
- $\text{grdim}(H^0(\check{\mathcal{M}}_n^d, L^k)) = \sum_{i=0}^{\infty} \dim(H^0(\check{\mathcal{M}}_n^d, L^k)^i) t^i \in \mathbb{Z}[[t]]$
- $\text{grdim}(H^0(\check{\mathcal{M}}_n^d, L^k)) = \chi_{\mathbb{T}}(\check{\mathcal{M}}_n^d, L^k) \in \widehat{K_{\mathbb{T}}(*)} \cong \widehat{R(\mathbb{T})} \cong \mathbb{Z}[[t^{\pm 1}]]$
- (Paradan 2011)  $\rightsquigarrow$

$$\chi_{\mathbb{T}}(\check{\mathcal{M}}_n^d, L^k) = \sum_{F_i} \int_{F_i} \text{ch}_{\mathbb{T}}(L^k|_{F_i} \otimes \text{Sym} N^* F_i) \text{Todd}(TF_i)$$

- $F_i \subset (\check{\mathcal{M}}_n^d)^{\mathbb{T}}$  fixed point components
- (Hausel–Szenes, 2003) direct computation  $\rightsquigarrow \chi_{\mathbb{T}}(\check{\mathcal{M}}_2^1, L^k) =$

$$\sum_{a=1, t, 1/t} \text{Res}_{z=a} \frac{\frac{2^{2g-1}}{(1-t)^{g-1}} \left[ k+1 + \frac{zt}{1-zt} + \frac{t/z}{1-t/z} \right]^g}{\left[ z^{k+1} \frac{1-t/z}{1-tz} - z^{-(k+1)} \frac{1-tz}{1-t/z} \right] [(1-1/z)(1-z)(1-t/z)(1-tz)]^{g-1}} \frac{dz}{z},$$

- (Hausel–Szenes, 2003) conjecture for higher  $n$
- recently (Halpern-Leistner 2016) and (Andersen–Gukov–Pei 2016) gave formulas for  $\chi_{\mathbb{T}}(\mathcal{M}_G, L^k)$  for general  $G$  building on the work of (Teleman–Woodward 2009)

# Equivariant Verlinde algebra for Higgs bundles

- (Gukov, Pei 2015)  $\rightsquigarrow \chi_{\mathbb{T}}(\check{\mathcal{M}}_n^d, L^k)$  arises from a 1+1D TQFT
- $E_\lambda := \chi_{\mathbb{T}}(T^*\mathcal{F}; L_\lambda) = t_\lambda(t)P_\lambda/\psi_t \in R(T_n)^{S_n}[[t]]$   

$$P_\lambda = \frac{\sum_{w \in S_n} (-1)^{\sigma(w)} z^{w(\lambda)} w(\Delta_t)}{\Delta_1 t_\lambda(t)} \in R(\mathrm{SU}_n)[t]$$
 Hall-Littlewood  

$$\Delta_t = z^\rho \prod_{\alpha \in \Phi^-} (1 - tz^\alpha); \psi_t = \prod_{\alpha \in \Phi} (1 - tz^\alpha); t_\lambda(t) = \sum_{w \in \mathrm{St}_{S_n}(\lambda)} t^{l(w)}$$
- $\lambda_m := (k+1, 1, \dots, 1, 0, \dots, 0) = (k+1)\omega_1 + \omega_2 + \dots + \omega_m$   
 form  $B_{\lambda_m} = \frac{\sum_{w \in S_n} (-1)^{\sigma(w)} z^{w(\lambda_m)} (1 - tz^{w(-\theta)}) w(\Delta_t)}{\Delta_1 t_\lambda(t)} \in R(T_n)^{S_n}[t]$   
 symmetric Bethe-Ansatz polynomial
- $Q\mathrm{Ver}_n^k := \overline{\mathbb{C}(t)} R(T_n)^{S_n}[t] / (B_{\lambda_1}, \dots, B_{\lambda_{n-1}})$
- $\langle E_\lambda, E_{\eta^\dagger} \rangle_t := \delta_{\lambda\eta} \tilde{t}_\lambda(t) (1-t)^{n-1}$ ,  $\tilde{t}_\lambda(t) = \sum_{w \in \mathrm{St}_{\check{S}_n^k}(\lambda)} t^{l(w)}$

## Theorem (Hausel–Szenes 2016)

$(E_\lambda)_{\lambda_1 \leq k}$  is a basis of  $Q\mathrm{Ver}_n^k$ , and  $(Q\mathrm{Ver}_n^k, \langle, \rangle_t)$  is a Frobenius algebra computing equivariant Verlinde formula

- $(Q\mathrm{Ver}_n^k, \langle, \rangle_t)_{t=0} \cong (\mathrm{Ver}_n^k, \langle, \rangle)$
- is  $(Q\mathrm{Ver}_n^k, \langle, \rangle_t)$  the fusion algebra of some deformation of  $\widehat{\mathfrak{sl}}_n$ ?

- we found that
- Frobenius algebra  $\mathbb{C}[\mathrm{GL}_n(\mathbb{F}_q)]^{\mathrm{GL}_n(\mathbb{F}_q)} \rightsquigarrow \mathrm{PH}(\mathcal{M}_{\mathrm{Dol}}; q, -1)$

## Problem

*Is there  $t$ -deformation of  $\mathbb{C}[\mathrm{GL}_n(\mathbb{F}_q)]^{\mathrm{GL}_n(\mathbb{F}_q)}$  computing  $\mathrm{PH}(\mathcal{M}_{\mathrm{Dol}}; q, t)$ ?*

- Frobenius  $\mathbb{C}$ -algebra  $(\mathrm{Ver}_n^k, \langle, \rangle) \cong R_k(\widehat{\mathfrak{sl}}_n)$  computes  $\chi(\check{\mathcal{N}}; L^k)$
- Frobenius  $\overline{\mathbb{C}(t)}$ -algebra  $(Q\mathrm{Ver}_n^k, \langle, \rangle_t)$  computes  $\chi_{\mathbb{T}}(\check{\mathcal{M}}_{\mathrm{Dol}}; L^k)$

## Problem

*Is there a  $t$ -deformation of  $R_k(\widehat{\mathfrak{sl}}_n)$  computing  $\chi_{\mathbb{T}}(\check{\mathcal{M}}_{\mathrm{Dol}}; L^k)$ ?*