Quiver representations and cohomology of Hitchin fibers

Tamás Hausel

Royal Society URF, University of Oxford
http://www.maths.ox.ac.uk/~hausel/talks.html

Representation theory of quivers
and finite dimensional algebras
Oberwolfach
February 2011

Kac's conjecture on quiver representations

- Γ quiver with vertices $I = (1, \ldots, l)$
- $\mathbb{K}$ field ($\mathbb{C}$ or $\mathbb{F}_q$), $\alpha \in \mathbb{N}^I$ a dimension vector
- a quiver representation is *absolutely indecomposable* if it cannot be written as a direct sum of non-trivial quiver representations over $\overline{\mathbb{K}}$
- $A_\Gamma(\alpha, q) := \left| \left\{ \begin{array}{c} \text{abs. indec. reps of } \Gamma \text{ over } \mathbb{F}_q \text{ of} \\ \text{dimension } \alpha, \text{ modulo isomorphism} \end{array} \right\} \right|$

Theorem (Kac, 1982)

$A_\Gamma(\alpha, q) \in \mathbb{Z}[q]$ *and is independent of the orientation of* Γ.

Conjecture (Kac, 1982)

$A_\Gamma(\alpha, q) \in \mathbb{N}[q]$, *i.e. the coefficients of* $A_\Gamma(\alpha, q)$ *are* ≥ 0.

Quiver varieties

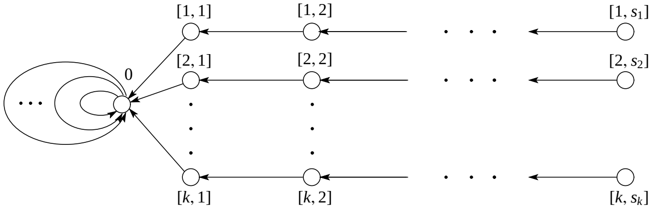
- fix Γ and $\alpha \in \mathbb{N}^I$
- $\mathcal{Q}^\alpha := T^*\mathrm{Rep}(\Gamma, \alpha)////_{(0,\zeta)}\mathrm{GL}_\alpha$
  an affine hyperkähler *quiver variety*
- when α indivisible (i.e. $gcd(\alpha_i) = 1$) $\zeta \in \mathbb{K}^I$ can be chosen so that $\mathcal{Q}^\alpha$ is non-singular

Theorem (Crawley-Boevey, Van den Bergh 2004)

α is indivisible $\rightsquigarrow$ Kac's conjecture true

$$P_c(\mathcal{Q}^\alpha_{\mathbb{C}}; \sqrt{q}) = \sum_i \dim(H_c^{2i}(\mathcal{Q}^\alpha_{\mathbb{C}}; \mathbb{Q}))q^i = q^{\dim} A_\Gamma(\alpha, q)$$

Comet-shaped quiver varieties



- Γ_μ comet-shaped quiver
- $\mu = (\mu^1, \ldots, \mu^k) \in \mathcal{P}(n)^{\{1,\ldots,k\}}$ s. t. $l(\mu^i) = s_i$
- when μ is indivis. $\mathcal{C}_j \subset \mathfrak{gl}(n, \mathbb{C})$ generic semisimple of type μ^j
- $\alpha_{[i,j]} := n - \sum_{r=1}^{j} \mu_r^i$ and $\alpha_0 := n$

Lemma

When μ is indivisible the hyperkähler quiver variety

$$\mathcal{Q}^{\alpha_\mu} \cong \{(A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_k)\, C_i \in \mathcal{C}_i |$$

$$A_1 B_1 - B_1 A_1 + \cdots + A_g B_g - B_g A_g + C_1 + \cdots + C_k = 0\} // \mathrm{GL}_n(\mathbb{C})$$

and is smooth.

Non-Abelian Hodge Theory

- Simpson (1990), Hitchin (1987) for Riemann surfaces
- G reductive complex algebraic group, M smooth complex projective variety
- Betti cohomology:

$$\mathcal{M}_{\mathrm{B}} := H^1_{\mathrm{B}}(M, \mathrm{G}) = \left\{ \begin{array}{c} \text{moduli space of representations} \\ \text{of } \pi_1(M) \to \mathrm{G} \end{array} \right\}$$

- De Rham cohomology:

$$\mathcal{M}_{\mathrm{DR}} := H^1_{\mathrm{DR}}(M, \mathrm{G}) = \{\text{moduli space of flat } \mathrm{G}\text{-connections on } M\}$$

- Delbeault cohomology:

$$\mathcal{M}_{\mathrm{Dol}} := H^1_{\mathrm{Dol}}(M, \mathrm{G}) = \{\text{moduli space of } \mathrm{G}\text{-Higgs bundles on } M\}$$

- Non-Abelian Hodge Theorem: $\mathcal{M}_{\mathrm{Dol}} \cong_{diff} \mathcal{M}_{\mathrm{DR}} \cong_{diff} \mathcal{M}_{\mathrm{B}}$

- $\mathrm{G} = \mathrm{GL}_n = \mathrm{GL}(n, \mathbb{C})$;
  $M = \Sigma_\mu$ coloured Riemann surface:
  - Σ compact Riemann surface with punctures
  - $a_1, \ldots, a_k \in \Sigma$ coloured by
  - $\mu = (\mu^1, \ldots, \mu^k) \in \mathcal{P}(n)^{\{1,\ldots,k\}}$ a partition of n at each puncture
- $\mathcal{M}^{\mu}_{\mathrm{Hit}} = \left\{ \begin{array}{c} \text{moduli space of solutions of} \\ \text{Hitchin self-duality equations on } \Sigma_\mu \end{array} \right\}$

  hyperkähler:
  $(\mathcal{M}^{\mu}_{\mathrm{Hit}}, I) \cong \mathcal{M}^{\mu}_{\mathrm{Dol}}$

  $(\mathcal{M}^{\mu}_{\mathrm{Hit}}, J) \cong (\mathcal{M}^{\mu}_{\mathrm{Hit}}, K) \cong \mathcal{M}^{\mu}_{\mathrm{DR}} \overset{RH}{\cong} \mathcal{M}^{\mu}_{\mathrm{B}}$

$$\begin{array}{rcl} RH : \mathcal{M}^{\mu}_{\mathrm{DR}} & \to & \mathcal{M}^{\mu}_{\mathrm{B}} \\ (E_\mu, \nabla) & \mapsto & \mathit{monodromy}(\nabla) \end{array}$$

Geometric aspects of NAHT for Riemann surfaces

- $\mathcal{M}^{\mu}_{\mathrm{Dol}} = \left\{ \begin{array}{c} \text{moduli space of } \mu\text{-parabolic rank } n \\ \text{Higgs bundles } (E_\mu, \phi) \text{ on } \Sigma \end{array} \right\}$

  the *Hitchin map*:
  $$\begin{array}{rcl} \chi : \mathcal{M}^{\mu}_{\mathrm{Dol}} & \to & \mathcal{A}^{\mu} \\ (E_\mu, \phi) & \mapsto & CharPol(\phi) \end{array}$$

  is a proper, completely integrable Hamiltonian system;
- $a \in \mathcal{A}^{\mu} \rightsquigarrow \Sigma_a \subset T^*\Sigma_\mu$ spectral curve; could be reducible, non-reduced;

Theorem (Beauville-Narasimhan-Ramanan 1989)

$\chi_\mu^{-1}(a) \cong \overline{\mathrm{Jac}}_\zeta(\Sigma_a)$ *certain moduli space of* ζ*-stable rank* 1 *torsion-free sheaves on* Σ_a*.*

- example: $0 \in \mathcal{A}_\mu \rightsquigarrow \Sigma_0 \subset T^*\Sigma_\mu$ non-reduced nth power of the zero section
- nilpotent cone: $\chi^{-1}(0) \cong \overline{\mathrm{Jac}}_\zeta(\Sigma_0)$

- $(\tilde{\mathcal{C}}_1, \dots, \tilde{\mathcal{C}}_k)$ semisimple conjugacy classes in GL_n of type μ.

$$\mathcal{M}_{\mathrm{B}}^{\mu} := \left\{ \begin{array}{c} A_1, B_1, ..., A_g, B_g \in \mathrm{GL}_n, C_1 \in \tilde{\mathcal{C}}_1, ..., C_k \in \tilde{\mathcal{C}}_k \\ [A_1, B_1] \cdots [A_g, B_g] C_1 \cdots C_k = I_n \end{array} \right\} // \mathrm{GL}_n$$

- $(\mathcal{C}_1, \dots, \mathcal{C}_k)$ semisimple adjoint orbits in $\mathfrak{gl}_n$ of type μ.

$$\mathcal{M}_{\mathrm{DR}}^{\mu} := \left\{ \begin{array}{c} \text{moduli space of meromorphic rank } n \text{ flat connections} \\ \text{with simple poles at the punctures and residue in } \mathcal{C}_i \end{array} \right\}$$

When $\Sigma = \mathbb{P}^1$ a point in

$$\mathcal{Q}^{\mu} := \{ C_1 \in \mathcal{C}_1, \dots, C_k \in \mathcal{C}_k | C_1 + \cdots + C_k = 0 \} // \mathrm{GL}_n$$

gives meromorphic flat connection $\sum_{i=1}^{k} C_i \frac{dz}{z - a_i} \in \mathcal{M}_{\mathrm{DR}}^{\mu}$ on $\mathbb{P}^1$

$$\Downarrow$$

$$\mathcal{Q}^{\mu} \quad \subset \quad \mathcal{M}_{\mathrm{DR}}^{\mu} \stackrel{RH}{\rightarrow} \mathcal{M}_{\mathrm{B}}^{\mu}$$

Weight filtration on cohomology

- X complex algebraic variety
- (Deligne 1974) constructs weight filtration

  $W_0 \subset \cdots \subset W_k \subset \cdots \subset W_{2d} = H_c^d(X;\mathbb{Q})$
- define $H_c(X;q,t) = \sum_{i,d} \dim(W_i/W_{i-1}(H_c^d(X)))t^d q^{i/2}$
- define $E(X;q) = H_c(X;q,-1)$
- W on $H_c^*(X,\mathbb{Q})$ is *pure* if $h^{i;d} = 0$ unless $i = d \Rightarrow$ $P_c(X;q^{1/2}) = E(X;q)$;
  examples: smooth projective varieties, $\mathcal{M}_{\mathrm{Dol}}$, $\mathcal{M}_{\mathrm{DR}}$, $\mathcal{Q}^\alpha$
  non-example: $\mathcal{M}_{\mathrm{B}}$
- in general the pure part of $H_c(X;q,t)$ is $PH_c(X;q) = \mathrm{Coeff}_{T^0}\left(H_c(X;qT^2,tT^{-1})\right)$; which, for a smooth X, is always the image of the cohomology of a smooth compactification

Perverse filtration on cohomology

- $f : X \to Y$, f projective morphism $\rightsquigarrow$
- (de Cataldo-Migliorini, 2005) construct perverse filtration: $P_0 \subset \cdots \subset P_i \subset \ldots P_k(X) \cong H^k(X)$
- for $f : X \to Y$ proper X smooth Y affine
  (de Cataldo-Migliorini, 2008):
  take $Y_0 \subset \cdots \subset Y_i \subset \ldots Y_d = Y$
  s.t. Y_i generic with $\dim(Y_i) = i$ then

$$P_{k-i-1}H^k(X) = \mathrm{ker}(\mathrm{H^k(X) \to H^k(f^{-1}(Y_i))})$$

- the Relative Hard Lefschetz Theorem holds:

$$\begin{array}{cccc} L^l : & Gr^P_{d-l}(H^*(X)) & \overset{\cong}{\to} & Gr^P_{d+l}H^{*+2l}(X) \\ & x & \mapsto & x \cup \alpha^l \end{array}$$

  where $\alpha \in W_2H^2(X)$ is a relative ample class
- example $\chi_\mu : \mathcal{M}^\mu_{\mathrm{Dol}} \to \mathcal{A}^\mu$ proper $\rightsquigarrow$ perverse filtration on $H^*(\mathcal{M}^\mu_{\mathrm{Dol}})$

Cohomological aspects of NAHT for Σ_μ

- Morse theory for $\mathbb{C}^\times \circlearrowright \mathcal{M}^\mu_{\mathrm{Dol}}$ by $(E_\mu, \phi) \mapsto (E_\mu, \lambda\phi)$

$$\Downarrow$$

$$H^*(\mathcal{M}^\mu_{\mathrm{Dol}}) \cong H^*(\chi_\mu^{-1}(0))$$

- Weight filtration is pure on $H^*(\mathcal{M}^\mu_{\mathrm{Dol}})$ and $H^*(\mathcal{M}^\mu_{\mathrm{DR}})$ but is not pure on $H^*(\mathcal{M}^\mu_{\mathrm{B}})$
- for $g = 0$ recall: $\mathcal{Q}^\mu \subset \mathcal{M}^\mu_{\mathrm{DR}} \overset{RH}{\to} \mathcal{M}^\mu_{\mathrm{B}}$
- *Purity Conjecture*:

$$PH^*_c(\mathcal{M}^\mu_{\mathrm{B}}) \cong H^*_c(\mathcal{Q}^\mu), \text{ if } \mu \text{ is indivisible.}$$
$$PH_c(\mathcal{M}^\mu_B, \sqrt{q}) = A_{\Gamma_\mu}(\alpha_\mu, q), \text{ if } \mu \text{ is divisible.}$$
$$\Downarrow$$

Kac's conjecture for star-shaped quivers Γ_μ

Cohomological aspects of NAHT for Σ_μ

- *$P = W$ Conjecture*: $P_k H^*(\mathcal{M}^\mu_{\mathrm{Dol}}) \cong W_{2k} H^*(\mathcal{M}^\mu_{\mathrm{B}})$
  Relative Hard Lefshetz $\rightsquigarrow$
- *Curious Poincaré Duality Conjecture:*

$$H^{p,p;k}(\mathcal{M}^\mu_{\mathrm{B}}) \cong H^{d_\mu - p, d_\mu - p; d_\mu + k - 2p}(\mathcal{M}^\mu_{\mathrm{B}})$$
$$\Downarrow$$
$$PH^*(\mathcal{M}^\mu_B) \cong H^{d_\mu}(\mathcal{M}^\mu_B)$$
$$\Downarrow$$
$$A_{\Gamma_\mu}(\alpha, 1) = b^{d_\mu}(\mathcal{M}^\mu_{\mathrm{B}})$$

- *Master Conjecture* with Macdonald polynomials $\tilde{H}_\lambda(\mathbf{x}_i; q, t)$

$$H_c(\mathcal{M}^\mu_{\mathrm{B}}; q, t) = (t\sqrt{q})^{d_\mu}(\tfrac{1}{q} - 1)(1 - qt^2)\cdot$$
$$\cdot \left\langle \mathrm{Log}\left(\sum_{\lambda \in \mathcal{P}} \left(\prod_{i=1}^k \tilde{H}_\lambda(\mathbf{x}_i; \tfrac{1}{q}, qt^2)\right) \mathcal{H}_\lambda(\tfrac{1}{q}, qt^2)\right), h_\mu \right\rangle$$

- $P = W$ when $n = 2$ by (de Cataldo–Hausel–Migliorini 2010)
- the pure part and the $t = -1$ specialization of the Master Conjecture are theorems of (Hausel–Letellier–Villegas; 2008)

Aspects of NAHT for Σ^μ

$$
\begin{array}{ccccccc}
 & & & (\mathcal{M}^\mu_{\mathrm{Hit}}, g) & & \\
 & & \swarrow & \downarrow & \searrow & \\
\chi_\mu^{-1}(0) \subset & \mathcal{M}^\mu_{\mathrm{Dol}} & \cong_{diff} & \mathcal{M}^\mu_{\mathrm{DR}} & \overset{RH}{\cong} & \mathcal{M}^\mu_{\mathrm{B}} \\
 & \chi_\mu \downarrow & & \cup\!\!\uparrow & \nearrow & \\
 & \mathcal{H}^\mu & & \mathcal{Q}^\mu & &
\end{array}
$$

- Purity Conjecture: $PH^*(\mathcal{M}^\mu_{\mathrm{B}}) \cong H^*(\mathcal{Q}^\mu)$
- $P = W$ Conjecture: $P_k H^*(\mathcal{M}^\mu_{\mathrm{Dol}}) \cong W_{2k} H^*(\mathcal{M}^\mu_{\mathrm{B}})$
- Master Conjecture: combinatorial formula for $H_c(\mathcal{M}_{\mathrm{B}}; q, t)$

Aspects of NAHT for Σ^μ

$$\begin{array}{cccccc} & & & (\mathcal{M}^\mu_{\mathrm{Hit}}, g) & & \\ & & \swarrow & \downarrow & \searrow & \\ \chi_\mu^{-1}(0)\subset & \mathcal{M}^\mu_{\mathrm{Dol}} & \cong_{diff} & \mathcal{M}^\mu_{\mathrm{DR}} & \overset{RH}{\cong} & \mathcal{M}^\mu_{\mathrm{B}} \\ & \chi_\mu \downarrow & & \cup\!\uparrow & \nearrow & \\ & \mathcal{H}^\mu & & \color{red}{\mathcal{Q}^\mu} & & \end{array}$$

- Purity Conjecture: $PH^*(\mathcal{M}^\mu_{\mathrm{B}}) \cong H^*(\mathcal{Q}^\mu)$
- $P = W$ Conjecture: $P_k H^*(\mathcal{M}^\mu_{\mathrm{Dol}}) \cong W_{2k} H^*(\mathcal{M}^\mu_{\mathrm{B}})$
- Master Conjecture: combinatorial formula for $H_c(\mathcal{M}^\mu_{\mathrm{B}}; q, t)$

$$
\begin{array}{ccccccc}
 & & & (\mathcal{M}^{\alpha}_{\mathrm{Hit}}, g) & & & \\
 & & \swarrow & \downarrow & \searrow & & \\
\chi_\alpha^{-1}(0) \subset & \mathcal{M}^{\alpha}_{\mathrm{Dol}} & \cong_{diff} & \mathcal{M}^{\alpha}_{\mathrm{DR}} & \overset{RH}{\cong} & \mathcal{M}^{\alpha}_{\mathrm{B}} & \\
 & \chi_\alpha \downarrow & & \cup\!\!\uparrow & \nearrow & & \\
 & \mathcal{H}^{\alpha} & & \mathcal{Q}^{\alpha} & & &
\end{array}
$$

- Purity Conjecture: $PH^*(\mathcal{M}^{\alpha}_{\mathrm{B}}) \cong H^*(\mathcal{Q}^{\alpha})$
- $P = W$ Conjecture: $P_k H^*(\mathcal{M}^{\alpha}_{\mathrm{Dol}}) \cong W_{2k} H^*(\mathcal{M}^{\alpha}_{\mathrm{B}})$
- Master Conjecture: combinatorial formula for $H_c(\mathcal{M}^{\alpha}_{\mathrm{B}}; q, t)$

Geometric ingredients to Graphical NAHT

- (Crawley-Boevey–Shaw, 2006): $(\Gamma, \alpha) \rightsquigarrow \mathcal{M}_B^\alpha$ *multiplicative quiver variety*, using group valued symplectic quotients of Alekseev-Malkin-Meinrenken. For a star-shaped quiver Γ_μ:

$$\mathcal{M}_B^{\alpha_\mu} \cong \mathcal{M}_B^\mu = \left\{ C_1 \in \tilde{\mathcal{C}}_1, \ldots, C_k \in \tilde{\mathcal{C}}_k | C_1 \cdots C_k = I \right\} // \mathrm{GL}_n$$

- (Crawley-Boevey 2011) $(\Gamma, \alpha) \rightsquigarrow$ moduli space of connections $\mathcal{M}_{\mathrm{DR}}^\alpha$ on an arrangement of Riemann surfaces based on the quiver and finds analogue of *RH*
- (Hausel-Proudfout 2007) constructs $\chi_\mu^{-1}(0)$ when $\alpha = (1, \ldots, 1) \rightsquigarrow$ for general $\alpha \in \mathbb{N}^i$ define $\chi_\mu^{-1}(0) := \overline{Jac_\zeta}(\Sigma_\alpha)$; where Σ_α is a connected projective plane curve with irreducible components $\{\Sigma_i\}_{i \in I}$ s.t
  1. Σ^{red} has only nodal singularities
  2. Σ_i^{red} is non-singular
  3. $mult(\Sigma_i) = \alpha_i$
- Is GNAHT just NAHT for the reduced nodal curve Σ_α^{red}?

$$
\begin{array}{ccccccc}
 & & & (\mathcal{M}^\alpha_{\mathrm{Hit}}, g) & & & \\
 & & \swarrow & \downarrow & \searrow & & \\
\chi_\alpha^{-1}(0) \subset & \mathcal{M}^\alpha_{\mathrm{Dol}} & \cong_{diff} & \mathcal{M}^\alpha_{\mathrm{DR}} & \overset{RH}{\cong} & \mathcal{M}^\alpha_{\mathrm{B}} & \\
 & \chi_\alpha \downarrow & & \uparrow\!\!\lrcorner & \nearrow & & \\
 & \mathcal{H}^\alpha & & \mathcal{Q}^\alpha & & &
\end{array}
$$

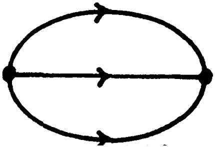
- Purity Conjecture: $PH^*(\mathcal{M}^\alpha_{\mathrm{B}}) \cong H^*(\mathcal{Q}^\alpha)$
- $P = W$ conjecture: $P_k H^*(\mathcal{M}^\alpha_{\mathrm{Dol}}) \cong W_{2k} H^*(\mathcal{M}^\alpha_{\mathrm{B}})$
- Master Conjecture: combinatorial formula for $H^\alpha_c(\mathcal{M}_{\mathrm{B}}; q, t)$

Graphical Non-Abelian Hodge Theory would

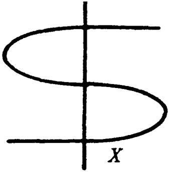
- unify NAHT with the theory of quiver varieties ($\rightsquigarrow$ representation theory of Kac-Moody algebras, Nakajima)

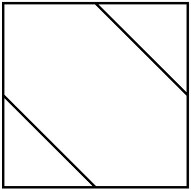
- describe the perverse filtration on the cohomology of the Hitchin fibers $\overline{\mathrm{Jac}}_\zeta(\Sigma_\alpha)$
$\overset{Ng\hat{o}}{\rightsquigarrow}$ orbital integrals on p-adic groups

- the Graphical Purity conjecture $PH_c(\mathcal{M}_B^\alpha, \sqrt{q}) = A_\Gamma(\alpha_\mu, q)$ would imply Kac's conjecture, that the coefficients of $A_\Gamma(\alpha_\mu, q)$ are non-negative

- quiver : 
- $\alpha = (1,1)$
- $\mathcal{Q}^\alpha \cong T^*\mathbb{P}^2 = \{\tilde{z}_1\tilde{w}_1 + \tilde{z}_2\tilde{w}_2 + \tilde{z}_3\tilde{w}_3 = 0\}//_\theta \mathbb{T}$
  $\mathcal{M}_{\mathrm{B}}^\alpha \cong mT^*\mathbb{P}^2 = \{(1+z_1w_1)(1+z_2w_2)(1+z_3w_3) = 1\}//_\theta \mathbb{T}$
  $\tilde{z}_i = z_i, \tilde{w}_1 = w_1, \tilde{w}_2 = (1+z_1w_1)w_2,$
  $\tilde{w}_3 = (1+z_1w_1)(1+z_2w_2)w_3$
  $(1+z_1w_1)(1+z_2w_2)(1+z_3w_3) = 1 \Rightarrow \tilde{z}_1\tilde{w}_1 + \tilde{z}_2\tilde{w}_2 + \tilde{z}_3\tilde{w}_3 = 0$
  $\Downarrow$
  $\imath_\alpha : \mathcal{M}_{\mathrm{B}}^\alpha \to \mathcal{Q}^\alpha$ algebraic embedding
- plus $RH : \mathcal{Q}^\alpha \to \mathcal{M}^\alpha$ induced from $(z_i, w_i) \mapsto (z_i, \frac{\exp(z_iw_i)-1}{z_i})$
- $\rightsquigarrow PH^*(\mathcal{M}_{\mathrm{B}}^\alpha) \cong H^*(\mathcal{Q}^\alpha)$

Example: Dollar sign curve

- $\Sigma_\alpha \cong$ 

$\overline{Jac}_\zeta(\Sigma_\alpha) \cong$ 
- $\chi_\alpha^{-1}(0)$ is three toric varieties glued together according to the toroidal hyperplane arrangement $\tilde{\mathcal{B}}$
- $b_4(\mathcal{M}^\alpha_{\mathrm{Dol}}) = \#\{$ 2-dimensional regions in $\tilde{\mathcal{B}}\} = 3 = \#\{$ vertices of $\mathcal{B}\} = b_0(T^*\mathbb{P}^2) + b_2(T^*\mathbb{P}^2) + b_4(T^*\mathbb{P}^2)$

$$\Downarrow$$

$H^4(\mathcal{M}^\alpha_{\mathrm{B}}) \cong H^*(\mathcal{Q}^\alpha)$
- $\sum_{p,k} h^{p;k}(\mathcal{M}^\alpha_{\mathrm{B}}) q^p t^k = 1 + 2qt + qt^2 + 2q^2t^2 + q^2t^4 + q^3t^4 + 2q^3t^3 + q^4t^4$