Quaternionic Geometry of Matroids

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”An electric circuit seemed to close; and a spark flashed forth the herald (as I foresaw immediately) of many long years to come of definitely directed thought and work by myself, if spared, and, at all events, on the part of others if I should even be allowed to live long enough distinctly to communicate the discovery. Nor could I resist the impulse - unphilosophical as it may have been - to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula which contains the Solution of the Problem, but, of course, the inscription has long since mouldered away.”
[William Rowan Hamilton in 1843, on his invention of quaternions]

Skew field of Quaternions

$$\mathbb{H} := \{a + bi + cj + dk | a, b, c, d \in \mathbb{R}\}$$

$$i^2 = j^2 = k^2 = ijk = -1$$

$$\Downarrow$$

$$ij = -ji = k$$
$$jk = -kj = i$$
$$ki = -ik = j$$

Hyperkähler manifolds

- A *hyperkähler manifold* is a Riemannian manifold of dimension $4n$ with holonomy contained in

$$Sp(n) \subset SU(2n) \subset U(2n) \subset SO(4n).$$

  Roughly speaking it is a Riemannian manifold, so that the metric is compatible with a quaternionic structure on the tangent bundle of the manifold
- Another definition is to say, that the manifold is Kähler with respect to three Kähler structures (I, ω_I, g), (J, ω_J, g) and (J, ω_K, g) corresponding to the same Riemannian metric g. Moreover the complex structures, as endomorphisms of the tangent bundle satisfy the quaternionic relationship

$$I^2 = J^2 = K^2 = IJK = -1.$$

  Singling out the complex structure I, the two form $\omega_J + i\omega_K$ becomes a *holomorphic symplectic* form.

Toric hyperkähler varieties

- $A = [a_1, a_2, \ldots, a_n]$, rationally represented matroid where $a_i \in \mathbb{Z}^d$

- $B = [b_1, \ldots, b_n]^T$ a Gale dual of A

$$0 \longrightarrow \mathbb{Z}^{n-d} \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^d \longrightarrow 0.$$

- $\mathbb{T}_{\mathbb{C}} = \mathbb{C}^{\times} = GL(1, \mathbb{C})$

$$1 \longleftarrow \mathbb{T}_{\mathbb{C}}^{n-d} \xleftarrow{B^T} \mathbb{T}_{\mathbb{C}}^n \xleftarrow{A^T} \mathbb{T}_{\mathbb{C}}^d \longleftarrow 1.$$

- $\mathbb{T}_{\mathbb{C}}^d \subset \mathbb{T}_{\mathbb{C}}^n \circlearrowright \mathbb{C}^n \Longrightarrow \mathbb{T}_{\mathbb{C}}^d \circlearrowright T^*\mathbb{C}^n \cong \mathbb{C}^n \times (\mathbb{C}^n)^*$

- $\mu : \mathbb{C}^n \times (\mathbb{C}^n)^* \to (\mathfrak{t}^d)^*$

  $\mu(v, w) = \sum_{i=1}^n v_i w_i a_i$. is the *holomorphic moment map*

- For $\xi \in (\mathfrak{t}_{\mathbb{C}}^d)^* \cong \mathbb{C}^d$

$$\mathcal{M}(\xi, A) = \mu^{-1}(\xi) // \mathbb{T}^d$$

  is the *affine toric hyperkähler variety*

Arithmetic Fourier Transform for $T^*\mathbb{C}P^n$

- Calabi's hyperkähler manifold with $A = [1, 1, \ldots, 1]$ $T^*\mathbb{C}P^n \cong \{(v,w) \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} | \sum_{i=1}^{n+1} v_i w_i = 1\}//\mathbb{T}$

- $f(\xi) = q\delta_0 + (q-1)1 =$

$$\#\{(v,w) \in \mathbb{F}_q \times \mathbb{F}_q | vw = \xi\} = \begin{cases} 2q-1 & \text{if } \xi = 0 \\ q-1 & \text{if } \xi \neq 0 \end{cases}$$

- $\frac{1}{q-1}\#\{(v,w) \in \mathbb{F}_q^{n+1} \times \mathbb{F}_q^{n+1} | \sum_i^n v_i w_i = 1\} =$

$$\frac{1}{q-1} f \star f \star \cdots \star f(1) =$$

$$\frac{q^{n/2}}{q-1} \sum_{X \in \mathbb{F}_q} \mathcal{F}(f)(X)^{n+1} \Psi(X) = \frac{q^{n/2}}{q-1}$$

$$\sum_{X \in \mathbb{F}_q} \left(q q^{-1/2} 1(X) + (q-1) q^{1/2} \delta_0(X) \right)^{n+1} \Psi(X)$$

$$= \frac{q^{2n+1} - q^n}{q-1} = q^n (q^n + q^{n-1} + \cdots + 1)$$

- By the Weil conjectures: $\Rightarrow P_t(T^*\mathbb{C}P^n) = 1 + t^2 + t^4 + \cdots + t^{2n}$

Arithmetic Fourier Transform for $\mathcal{M}(\xi, A)$

- $L(A)$ the intersection lattice of the hyperplane arrangement given by A
- $\mu_{L(A)}$ its Möbius function,
- $ca(V)$ the number of coatoms, i.e. hyperplanes containing $V \in L(A)$

Theorem 1 (Hausel, 2005).

$$\#(\mathcal{M}(\xi, A)(\mathbb{F}_q)) = \frac{q^{n-d}}{(q-1)^d} \sum_{V \in L(A)} \mu_{L(A)}(V) q^{ca(V)}$$

$\Downarrow$ *Weil conjectures*

$$P_t(\mathcal{M}(\xi, A)) = \frac{1}{t^{2d}(1-t^2)^d} \sum_{V \in L(A)} \mu_{L(A)}(V) (t^2)^{n-ca(V)}$$

Corollary 2 (Bielavksi-Dancer 2000, Hausel-Sturmfels 2002).

$$P_t(\mathcal{M}(\xi, A)) = \sum_{i=0}^{n-d} h_i(B) t^{2i}$$

the h-polynomial or reliability polynomial of the dual matroid B

An Application

- $B = [b_1, b_2, \ldots, b_n]$ a vector configuration
- $\mathcal{M}_\mathsf{B} = \{V \subset B | V$ is linearly independent$\}$ the corresponding matroid complex
- $f_i(B) = |\{V \in \mathcal{M}_\mathsf{B} || V| = i + 1\}|$ face number
- $h_i(B) = \sum_{j=0}^{i} (-1)^{i-j} \binom{k-j}{i-j} f_{j-1}(B)$ h-number

Theorem 3 (Hausel-Sturmfels 2002). *The cohomology $H^*(\mathcal{M}(\xi, A), \mathbb{C})$ satisfies the injectivity part of the Hard Lefschetz theorem.*

Corollary 4 (Hausel-Sturmfels 2002, Swartz 2003, Hausel 2004). *The h-vector of a (rationally representable) matroid satisfies the g-inequalities of McMullen.*

"In general, although in one sense I hope that I am actually growing modest about the quaternions, from my seeing so many peeps and vistas into future expansions of their principles, I still must assert that this discovery appears to me to be as important for the middle of the nineteenth century as the discovery of fluxions was for the close of the seventeenth."
[William Rowan Hamilton in 1853, ten years after his discovery of quaternions]