Positivity for Kac polynomials and DT-invariants of quivers

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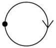
Quivers and their representations

- a *quiver* Γ is an oriented and connected graph with vertices $I = (1, \ldots, n)$ and arrows or oriented edges $E \subset I \times I$, (possibly multiple edges and edge-loops)
- $\mathbb{K}$ field; (either $\mathbb{C}$, $\mathbb{F}_q$ or $\overline{\mathbb{F}}_q$)
- a *representation* ρ of Γ is a collection of finite dimensional $\mathbb{K}$-vector spaces $\{V_i\}_{i \in I}$ and homomorphisms $\rho_a \in \mathrm{Hom}(V_i, V_j)$ for every $a = (i, j) \in E$
- $\dim \rho = (\dim V_1, \ldots, \dim V_n) \in \mathbb{N}^I$ is the *dimension* of ρ
- every representation can be written uniquely as a sum of *indecomposable* representations
- example: representations of $S_1 :=$  (matrices up to conjugation) are classified by Jordan normal form.
- for algebraically closed $\mathbb{K}$ indecomposables are $\begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \ddots & 0 & 0 \\ 0 & 0 & \lambda & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}$

The Kac polynomial

- $\alpha \in \mathbb{N}^I$ a dimension vector
- a quiver representation is *absolutely indecomposable* if it cannot be written as a direct sum of non-trivial quiver representations over $\overline{\mathbb{K}}$
- $A_\Gamma(\alpha, q) := \left| \left\{ \begin{array}{c} \text{abs. indec. reps of } \Gamma \text{ over } \mathbb{F}_q \text{ of} \\ \text{dimension } \alpha\text{, modulo isomorphism} \end{array} \right\} \right|$

Theorem (Kac, 1982)

$A_\Gamma(\alpha, q) \in \mathbb{Z}[q]$ *and is independent of the orientation of* Γ*.*

- example: $A_{S_1}(n, q) = q$

Kac's conjectures

Conjecture (Kac, 1982)

1. *When Γ is loopless, the constant term $A_\Gamma(\alpha, 0) = m_\alpha$*
2. $A_\Gamma(\alpha, q) \in \mathbb{N}[q]$ *, i.e. the coefficients of $A_\Gamma(\alpha, q)$ are ≥ 0.*

Theorem (Crawley-Boevey,Van den Bergh 2004)

Both conjectures hold for any quiver with α indivisible; i.e. $gcd(\alpha(i)) = 1$

Theorem (Hausel, 2010)

Kac's Conjecture 1 holds for every quiver and dimension vector.

Theorem (Mozgovoy, 2012)

Kac's Conjecture 2 holds for quivers with enough loops but any dimension vector.

Symplectic Quiver varieties

- quiver Γ; let $\{V_i\}_{i\in I}$ collection of vector spaces of dim $= \alpha \in \mathbb{N}^I$
- $\mathbb{V}_\alpha = \bigoplus_{(i,j)\in E} \mathrm{Hom}(V_i, V_j)$
- $\mathrm{G}_\alpha = \bigtimes_{i\in I} \mathrm{GL}(V_i)/\mathrm{GL}_1$, $\mathfrak{g}_\alpha = \{X_i \in \mathfrak{gl}(\mathbb{V}_i) | \sum_i \mathrm{tr}(X_i) = 0\}$
- action $\rho : \mathrm{G}_\alpha \to \mathrm{GL}(\mathbb{V}_\alpha)$, with derivative $\varrho : \mathfrak{g}_\alpha \to \mathfrak{gl}(\mathbb{V}_\alpha)$
- *moment map* $\mu_\alpha : \mathbb{V}_\alpha \times \mathbb{V}_\alpha^* \to \mathfrak{g}_\alpha^*$ by $\mu(v, w)(X) = \langle \varrho(X)v, w\rangle$
- for a *generic* $\xi \in (\mathfrak{g}_\alpha^*)^{\mathrm{G}_\alpha}$ *quiver variety* $\mathcal{M}_\alpha := \mu_\alpha^{-1}(\xi)//\mathrm{G}_\alpha$
- example: for $\Gamma = S_1$ $\mathcal{M}_\alpha$ is the commuting variety
- if $\alpha \in \mathbb{N}^I$ is indivisible ($\gcd(\alpha) = 1$) then $\mathcal{M}_\alpha$ is non-singular, if α is divisible ($\gcd(\alpha) > 1$) $\mathcal{M}_\alpha$ has singular points
- (Crawley-Boevey, Van den Bergh 2004) when α indivisible $|\mathcal{M}_\alpha(\mathbb{F}_q)| = q^{d_\alpha} A_\Gamma(\alpha, q)$ & $H_c^*(\mathcal{M}_\alpha; \mathbb{C})$ is pure $\rightsquigarrow$

$$A_\Gamma(\alpha, q) = \sum_k \dim(H_c^{2k}(\mathcal{M}_\alpha/\mathbb{C}, \mathbb{C}))q^{k-d_\alpha} \in \mathbb{N}[q]$$

$\rightsquigarrow$ Kac's Conjecture 2 when α indivisible

Weyl group action on regular semisimple adjoint orbits

- G_α our connected complex reductive group, $X \in \mathfrak{g}_\alpha (\cong \mathfrak{g}_\alpha^*)$ regular semisimple $\Leftrightarrow C_{\mathrm{G}_\alpha}(X) = \mathrm{T}_\alpha$, its (co)adjoint orbit $\mathcal{O}_\alpha \cong \mathrm{G}_\alpha/\mathrm{T}_\alpha \subset \mathfrak{g}_\alpha$
- $\rightsquigarrow$ Weyl group $W_\alpha := N_{\mathrm{G}_\alpha}(\mathrm{T}_\alpha)/\mathrm{T}_\alpha$ acts on $\mathcal{O}_\alpha$
  $\rightsquigarrow$ G_α acts on $H_c^*(\mathcal{O}_\alpha, \mathbb{C})$
- $H_c^*(\mathcal{O}_\alpha, \mathbb{C}) = \bigoplus_{V_\chi \in \mathrm{Irr}(W_\alpha)} H_c^*(\mathcal{O}_\alpha, \mathbb{C})_\chi \otimes V_\chi$,
- incidently, $\mathcal{O}_\alpha$ is a quiver variety, by purity

$$\sum_k \dim(H_c^{2k}(\mathcal{O}_\alpha, \mathbb{C})_\chi) q^k = \sum_{w \in W_\alpha} \frac{\chi(w)}{|W_\alpha|} \sum_k \mathrm{Tr}(\mathrm{Frob}_q \circ w, H_c^{2k}(\mathcal{O}_\alpha/\overline{\mathbb{F}}_q, \mathbb{Q}_\ell))$$
$$= \sum_{w \in W_\alpha} \frac{\chi(w)}{|W_\alpha|} |\mathcal{O}_\alpha^w(\mathbb{F}_q)| = \sum_{w \in W_\alpha} \frac{\chi(w)}{|W_\alpha|} \frac{|\mathrm{G}_\alpha(\mathbb{F}_q)|}{|T_\alpha^w|}$$

- $\rightsquigarrow$ for $\epsilon : W_\alpha \to \{\pm 1\} \subset \mathbb{C}^\times$ *sign character*:

$$\dim(H_c^k(\mathcal{O}_\alpha, \mathbb{C})_\epsilon) = \begin{cases} 1 & k = \dim(\mathcal{C}) \\ 0 & ow \end{cases}$$

Weyl group action on cohomology of quiver varieties

- $\mathcal{O}_\alpha \in \mathfrak{g}_\alpha^*/\mathrm{G}_\alpha$ generic regular semisimple coadjoint orbit
- $\mathcal{M}_{\tilde{\alpha}} := \mu_\alpha^{-1}(\mathcal{O}_\alpha)//\mathrm{G}_\alpha$ is always non-singular
- $\mathcal{M}_{\tilde{\alpha}}$ is a quiver variety for an extended quiver $\tilde{\Gamma}$ and $\tilde{\alpha} \in \mathbb{N}^{\tilde{I}}$
- (Nakajima 1994-2003, Lusztig 2000, Maffei 2002)
  $W_\alpha := N_{\mathrm{G}_\alpha}(\mathrm{T}_\alpha)/\mathrm{T}_\alpha$ acts on $H_c^*(\mathcal{M}_{\tilde{\alpha}}, \mathbb{C})$

Theorem (Hausel–Letellier–Villegas 2012)

For the sign character $\epsilon : W_\alpha \to \{\pm 1\} \subset \mathbb{C}^\times$:

$$A_\Gamma(\alpha, q) = \sum_k \dim(H_c^{2k}(\mathcal{M}_{\tilde{\alpha}}, \mathbb{C})_\epsilon) q^{k-d_\alpha}$$

Corollary (Hausel–Letellier–Villegas 2012)

Kac's Conjecture 2 is valid for all quivers and all dimension vectors:

$$A_\Gamma(\alpha, q) \in \mathbb{N}[q]$$

Proof by arithmetic Fourier transform

- by purity and some arithmetic and geometric arguments:

$$\sum_k \dim(H_c^{2k}(\mathcal{M}_{\tilde{\alpha}},\mathbb{C})_\chi)q^k = \sum_{w\in W_\alpha}\frac{\chi(w)}{|W_\alpha|}\sum_k \mathrm{Tr}(\mathrm{Frob}_q\circ w, H_c^{2k}(\mathcal{M}_{\tilde{\alpha}}/\overline{\mathbb{F}}_q,\mathbb{Q}_\ell))$$

$$= \sum_{w\in W_\alpha}\frac{\chi(w)}{|W_\alpha|}|\mathcal{M}^w_{\tilde{\alpha}}(\mathbb{F}_q)| = \sum_{w\in W_\alpha}\frac{\chi(w)}{|W_\alpha|}\frac{|\mu_\alpha^{-1}(\mathcal{O}^w_\alpha)(\mathbb{F}_q)|}{|\mathrm{G}_\alpha(\mathbb{F}_q)|}$$

$\mathcal{M}^w_{\tilde{\alpha}} := \mu_\alpha^{-1}(\mathcal{O}^w_\alpha)//\mathrm{G}_\alpha$, where $\mathcal{O}^w_\alpha := (\mathcal{O}_\alpha/\overline{\mathbb{F}}_q)^{Frob_q\circ w}$

- evaluate $|\mu_\alpha^{-1}(\mathcal{O}^w_\alpha)(\mathbb{F}_q)|$ by arithmetic Fourier transform
- fix $\Psi : \mathbb{F}_q \to \mathbb{C}^\times$ non-trivial additive character
- for $f : \mathfrak{g}_\alpha(\mathbb{F}_q) \to \mathbb{C}$ define $\mathcal{F}(f) : \mathfrak{g}^*_\alpha(\mathbb{F}_q) \to \mathbb{C}$ by

$$\mathcal{F}(f)(Y) := \sum_{X\in\mathfrak{g}_\alpha(\mathbb{F}_q)} f(X)\Psi(\langle X, Y\rangle).$$

- $|\mu_\alpha^{-1}(\mathcal{O}^w_\alpha)(\mathbb{F}_q)| \overset{\text{(Hausel,2010)}}{=} \frac{|\mathbb{V}_\alpha(\mathbb{F}_q)|}{|\mathfrak{g}_\alpha(\mathbb{F}_q)|}\sum_{X\in\mathfrak{g}_\alpha(\mathbb{F}_q)} |\ker(\varrho(X))|\mathcal{F}(\mathcal{O}^w_\alpha)(X)$

  recall $\varrho : \mathfrak{g}_\alpha \to \mathfrak{gl}(\mathbb{V}_\alpha)$ is the infinitesimal action

Master formula

- we use (Letellier 2005) to evaluate $\mathcal{F}(\mathcal{O}_\alpha^w)(X) \rightsquigarrow$

Theorem (Hausel–Letellier–Villegas 2012≤)

$$\frac{1}{q-1}\sum_{\chi^\mu\in W_\alpha}\sum_k \dim(H_c^{2k}(\mathcal{M}_{\tilde{\alpha}},\mathbb{C}))_{\chi^\mu} q^{k-d_\alpha} s_\mu(\mathbf{x}) =$$

$$= \mathrm{Log}\left(\sum_{\pi=(\pi^1,\dots,\pi^r)\in\mathcal{P}^r} \frac{\prod_{(i,j)\in\Omega} q^{\langle\pi^i,\pi^j\rangle}}{\prod_{i\in I} q^{\langle\pi^i,\pi^i\rangle}\prod_k\prod_{j=1}^{m_k(\pi^i)}(1-q^{-j})}\prod_{i=1}^r \tilde{H}_{\pi^i}(\mathbf{x}_i;q)\right)$$

where $\mathbf{x}_i = (x_{i,1}, x_{i,2}, \dots)$ *is a set of infinitely many independent variables;*$s_\mu(\mathbf{x}) = s_{\mu^1}(\mathbf{x}_1)\cdot\dots\cdot s_{\mu^n}(\mathbf{x}_n)$ *product of Schur functions;* $\tilde{H}_{\pi^i}(\mathbf{x}_i;q)$ *denote the Hall-Littlewood polynomial, which is a symmetric function in* $\mathbf{x}_i$ *and polynomial in* q*. Finally* Log *is the pletysthic logarithm.*

Refined Donaldson–Thomas invariants

- for any Calabi–Yau 3-fold X (Kontsevich–Soibelman, 2008) attempts to associate refined Donaldson–Thomas invariants , depending only on the 3-Calabi–Yau category $D^b Coh(X)$
- the construction is complete for the 3-Calabi–Yau categories associated to a quiver Γ with a potential
- let $\mathrm{DT}_\alpha(q) \in \mathbb{Z}[q]$ denote the refined Donaldson–Thomas invariants corresponding to Γ and α with 0 potential and 0 stability condition
- $\mathrm{DT}_\alpha(q) \in \mathbb{N}[q]$ was conjectured by (Kontsevich–Soibelman, 2010) and proved by (Efimov, 2011)

Corollary (Hausel–Letellier–Villegas 2012≤)

$$\sum_k \dim(H_c^{2k}(\mathcal{M}_{\tilde{\alpha}}, \mathbb{C})^{W_\alpha}) q^{k-d_\alpha} = \mathrm{DT}_\alpha(q) \in \mathbb{N}[q]$$

Stringy cohomology of linear symplectic quotients

- G complex reductive group $\rho : \mathrm{G} \to \mathrm{GL}(\mathbb{V})$ rep $\varrho : \mathfrak{g} \to \mathfrak{gl}(\mathbb{V})$
- moment map $\mu : \mathbb{V} \times \mathbb{V}^* \to \mathfrak{g}^*$ by $\mu(v,w)(X) = \langle \varrho(X)v, w \rangle$
- $Y := \mathbb{V} \times \mathbb{V}^* ////\mathrm{G} = \mu^{-1}(0)//\mathrm{G}$
- $\mathcal{C} \subset \mathfrak{g}^*$ *generic* regular semisimple coadjoint orbit
- $\tilde{Y} := \mu^{-1}(\mathcal{C})//\mathrm{G}$ is an orbifold
- W acts on $H^*_{str}(\tilde{Y}, \mathbb{C}) := \bigoplus_{(g)} H^{*-F(g)}(\tilde{Y}_{(g)})$
- ϵ sign representation of W on $H^{mid}(\mathcal{C})$
- $H^*_{str}(Y) := H^*_{str}(\tilde{Y}; \mathbb{C})_\epsilon$ module over $H^*_{BPS}(Y) := H^*_{str}(\tilde{Y}; \mathbb{C})^W$
- $A_\rho(q) := \sum_k \dim(H^{d-2k}_{str}(Y))q^k$
- $DT_\rho(q) := \sum_k \dim(H^{d-2k}_{BPS}(Y))q^k$
- questions:
  1. intrinsic definition?
  2. what is 3-CY category?
  3. can $D^b Coh_{str}(Y)$ be defined?
  4. multiplicative analogue?
  5. Higgs analogue?