Mixed Hodge polynomials of character varieties

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Character spaces

- Σ closed Riemann surface of genus g
- G a (compact Lie, quantum or finite) group
- $$\begin{aligned} \mathcal{M}(\Sigma, \mathsf{G}): &= \mathsf{Hom}(\pi_1(\Sigma), \mathsf{G})/\mathsf{G} \\ &= \{A_1, B_1, \ldots, A_g, B_g \in \mathsf{G} | \\ &\quad [A_1, B_1] \cdot \cdots \cdot [A_g, B_g] = 1\}/\mathsf{G} \end{aligned}$$
- measure on $\mathcal{M}(\Sigma, \mathsf{G})$ is
  - the Liouville measure of the Atiyah-Bott symplectic form, when G is a compact group
  - is a non-commutative quantization of this when G is a quantum group,
  - by measuring a representation P with weight $1/|\mathsf{Aut}(P)|$ for a finite group G.

Partition function of Riemann surfaces

- when G compact group (Witten 1991)

$$Z_{YM}(\Sigma, \mathrm{G}) \;=\; Vol(\mathcal{M}(\Sigma, \mathrm{G})) = \sum_{\chi \in Irr(\mathrm{G})} \left(\frac{|\mathrm{G}|}{\mathrm{dim}(\chi)} \right)^{2g-2}$$

e.g. $Z_{YM}(\Sigma, \mathrm{SU}(2))$ Riemann zeta function

- when G quantum group (Rouchet-Szenes 2000)

$$Z_{YM_q}(\Sigma, \mathrm{G}) \;=\; Vol(\mathcal{M}(\Sigma, \mathrm{G})) = \sum_{\chi \in Irr(\mathrm{G})} \left(\frac{|\mathrm{G}|_q}{\mathrm{dim}_q(\chi)} \right)^{2g-2}$$

is the Verlinde formula

- when G finite group (Freed-Quinn 1993)

$$Z_{CS}(\Sigma, \mathrm{G}) \;=\; Vol(\mathcal{M}(\Sigma, \mathrm{G})) = \sum_{\chi \in Irr(\mathrm{G})} \left(\frac{|\mathrm{G}|}{\mathrm{dim}(\chi)} \right)^{2g-2}$$

$\mathcal{M}(\Sigma, S_n) \rightsquigarrow$ Hurwitz spaces, $Z_{CS}(\Sigma, S_n) \rightsquigarrow$ Gromov-Witten theory.

What is the geometrical meaning of $Z_{CS}(\Sigma, \mathrm{GL}_n(\mathbb{F}_q))$?

Mixed Hodge Structure of Deligne

- $\bigoplus_{p,q} H_c^{p,q;k}(M)$ is the associated graded to the weight and Hodge filtrations on the compactly supported cohomology $H_c^k(M,\mathbb{C})$ of a complex algebraic variety M

- $h_c^{p,q;k} = \dim(H_c^{p,q;k}(M))$, the compactly supported *mixed Hodge numbers*

- $H_c(M;x,y,t) = \sum_{p,q,k} h_c^{p,q;k}(M)x^p y^q t^k$, the *mixed Hodge polynomial*

- $P_c(M;t) = H_c(M;1,1,t)$, the *Poincaré polynomial*

- $E(M;x,y) = H(x,y,-1)$, the *E-polynomial* of a smooth variety M.

Arithmetic and topological content of the E-polynomial

Theorem 1 (Katz 2006). *If M is a variety defined over $\mathbb{Z}$ and*

$$\#\{M(\mathbb{F}_q)\} = E(q)$$

is a polynomial in q, then

$$E(M; x, y) = E(xy).$$

- MHS on $H_c^*(M, \mathbb{C})$ is *pure* if $h_c^{p,q;k} = 0$ unless $p + q = k \Leftrightarrow H_c(M; x, y, t) = E(-xt, -yt) \Rightarrow P_c(M; t) = H_c(M; 1, 1, t) = E(-t, -t)$; examples of varieties with pure MHS: smooth projective varieties, $\mathcal{M}_{\mathrm{Dol}}$, $\mathcal{M}_{\mathrm{DR}}$, quiver varieties
- in general the pure part of $H_c(M; x, y, t)$ is $PH_c(M; x, y) = \mathrm{Coeff}_{T^0}\Big(H(M; xT, yT, tT^{-1})\Big)$; which, for a smooth M, is always the surviving part in the cohomology of a smooth compactification

First approximation

Let $\mathsf{G} = \mathsf{GL}_n(\mathbb{C})$, and consider the character stack

$$\begin{aligned}\mathcal{M}(\Sigma, \mathsf{G}) &= \{A_1, B_1, \ldots, A_g, B_g \in \mathsf{G} | \\ &\quad [A_1, B_1] \cdot \cdots \cdot [A_g, B_g] = 1\}/\mathsf{G}.\end{aligned}$$

First approximation

$$\begin{aligned}E(\mathcal{M}(\Sigma, \mathsf{G}); q) &= \#\mathcal{M}(\Sigma, \mathsf{GL}_n(\mathbb{F}_q)) \\ &= \sum_{\chi \in \mathrm{Irr}(\mathrm{GL_n}(\mathbb{F}_q))} \frac{|\mathsf{GL}_n(\mathbb{F}_q)|^{2g-2}}{\dim(\chi)^{2g-2}} \\ &= Z_{CS}(\Sigma, \mathsf{GL}_n(\mathbb{F}_q)).\end{aligned}$$

Smooth character varieties

- Coloured Riemann surface: $a_1, \ldots, a_k \in \Sigma$; put $k > 0$ partitions $\mu \in \mathcal{P}(n)^{\{1,..,k\}}$ at the punctures; $\tilde{\mathcal{C}}_i \subset \mathsf{GL}_n(\mathbb{C})$ semisimple conjugacy classes of type $\mu^i \in \mathcal{P}(n)$; denote such coloured Σ by Σ_μ.

- $\mathcal{M}(\Sigma_\mu, \mathsf{GL}_n) :=$
$$= \{(A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_k)\, C_i \in \tilde{\mathcal{C}}_i | \\ A_1^{-1}B_1^{-1}A_1B_1 \ldots A_g^{-1}B_g^{-1}A_gB_gC_1 \ldots C_k = Id\}//\mathsf{GL}_n$$

- Taking generic eigenvalues for $\tilde{\mathcal{C}}_i$ the character variety can always be made smooth with a given g and μ

- For example, when $k = 1$ and $\mu^1 = (n)$, the generic character variety is smooth:

$$\mathcal{M}(\Sigma_{(n)}, \mathsf{GL}_n) = \{(A_1, B_1, \ldots, A_g, B_g)\,| \\ A_1^{-1}B_1^{-1}A_1B_1 \ldots A_g^{-1}B_g^{-1}A_gB_g = \zeta_n Id\}//\mathsf{GL}_n,$$

where ζ_n is a primitive nth root of unity.

Second Approximation

$$\begin{aligned} H_c(\mathcal{M}(\Sigma_\mu, \mathsf{GL}_n); q, -1) &= E(\mathcal{M}(\Sigma_\mu, \mathsf{GL}_n); q) \\ &= \#\mathcal{M}(\Sigma_\mu, \mathsf{GL}_n(\mathbb{F}_q)) \end{aligned}$$

$$= (q-1) \sum_{\chi \in \mathsf{Irr}(\mathsf{GL_n}(\mathbb{F}_\mathsf{q}))} \frac{|\mathsf{GL}_n(\mathbb{F}_q)|^{2g-2}}{\mathsf{dim}(\chi)^{2g-2}} \prod_i^k \frac{\chi(\tilde{C}_i)}{\chi(1)} |\tilde{C}_i|$$

Character formula for $H_c(\mathcal{M}(\Sigma_\mu, \mathsf{GL}_n); q, t)$?

Quiver varieties

- $\mathcal{C}_i$ semisimple adjoint orbit in $\mathfrak{gl}_n(\mathbb{C})$ of type μ^i
- $\mathcal{M}(\Sigma_\mu, \mathfrak{gl}_n) :=$ $\{(A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_k)\, C_i \in \mathcal{C}_i |$ $\sum_{j=1}^g A_j B_j - B_j A_j + \sum_{i=1}^k C_i = 0\} // \mathrm{GL}_n(\mathbb{C})$, is a crab-shaped quiver variety
- if $gcd(\mu) = 1$ $\mathcal{M}(\Sigma_\mu, \mathfrak{gl}_n)$ is generically smooth
- when $g = 0$ a point in $\mathcal{M}(\mathbb{P}^1_\mu, \mathfrak{gl}_n)$ gives the meromorphic flat $\mathrm{GL}_n(\mathbb{C})$-connection $\sum C_i \frac{dz}{z-a_i}$ on the trivial bundle on $\mathbb{P}^1$.
- $\tilde{\mathcal{C}}_i = \exp(2\pi i \mathcal{C}_i) \subset G(\mathbb{C})$ is the corresponding conjugacy class
- the *Riemann-Hilbert monodromy map*

$$\nu^a : \mathcal{M}(\Sigma_\mu, \mathfrak{gl}_n) \to \mathcal{M}(\Sigma_\mu, \mathrm{GL}_n)$$

  is given by sending the flat connection to its holonomy.

Purity conjecture

Conjecture 1. *(Hausel 2005) 1. If $g = 0$ and $gcd(\alpha) = 1$ and $\mathcal{C}_i$ are generic, then*

$$\nu_*^a : H_c^*(\mathcal{M}(\Sigma_\mu, \mathfrak{gl}_n)) \xrightarrow{\cong} PH_c^*(\mathcal{M}(\Sigma_\mu, \mathsf{GL}_n))$$

2. For any $g \geq 0$ and $gcd(\mu) = 1$ and $\mathcal{C}_i$ are generic, then

$$PH_c(\mathcal{M}(\Sigma_\mu, \mathsf{GL}_n); q) = E(\mathcal{M}(\Sigma_\mu, \mathfrak{gl}_n); q)$$

Third approximation

$$\begin{aligned} PH_c(\mathcal{M}(\Sigma_\mu, \mathsf{GL}_n); q) &= E(\mathcal{M}(\Sigma_\mu, \mathfrak{gl}_n); q) \\ &= \#\mathcal{M}(\Sigma_\mu, \mathfrak{gl}_n(\mathbb{F}_q)) \end{aligned}$$

$$= \frac{|\mathfrak{gl}_n(\mathbb{F}_q)|^{g-1}}{|\mathsf{PGL}_n(\mathbb{F}_q)|} \sum_{x \in \mathfrak{gl}_n(\mathbb{F}_q)^*} |C_{\mathfrak{gl}_n}(x)|^g \prod_{i=1}^{k} \mathcal{F}(1_{\mathcal{C}_i})(x),$$

where $C_{\mathfrak{gl}_n}(x)$ is the centralizer of x under the coadjoint action of $\mathfrak{gl}_n$ on $\mathfrak{gl}_n^*$, and

$$\mathcal{F}(1_{\mathcal{C}_i})(x) := |\mathfrak{g}|^{-1/2} \sum_{y \in \mathfrak{gl}_n} 1_{\mathcal{C}_i}(y) \Psi(\langle y, x \rangle),$$

where $\Psi : \mathbb{F}_q \to \mathbb{C}^*$ is a fixed non-trivial additive character.

Character formula for $H_c(\mathcal{M}(\Sigma_\mu, \mathsf{GL}_n); q, t)$?

Macdonald polynomials

- $\mathbf{x} = \{x_1, x_2, \dots\}$ and $\mathbf{y} = \{y_1, y_2, \dots\}$; $\Lambda(\mathbf{x})$ and $\Lambda(\mathbf{y})$ rings of symmetric functions

- for a $\lambda \in \mathcal{P}$ partition, define Macdonald symmetric functions $\tilde{H}_\lambda(\mathbf{x}; q, t) \in \mathbb{Q}(q,t)[\Lambda(\mathbf{x})]$ by the *triangular property*

$$\tilde{H}_\lambda[X(1-t)] \in \mathbb{Q}(q,t)\{s_\nu | \nu \geq \lambda'\}$$

and the *Cauchy formula*:

$$\operatorname{Exp}\left(\frac{m_{(1)}(\mathbf{x}) m_{(1)}(\mathbf{y})}{(q-1)(1-t)}\right) = \sum_{\lambda \in \mathcal{P}} \tilde{H}_\lambda(\mathbf{x}; q, t) \tilde{H}_\lambda(\mathbf{y}; q, t) \mathcal{H}_\lambda(q, t),$$

where

$$\mathcal{H}_\lambda(q,t) := \prod \frac{1}{(q^{a+1} - t^l)(q^a - t^{l+1})}$$

Generalized Cauchy formula

- $\mathbf{x}_1 = \{x_{1,1}, x_{1,2}, \ldots\}, \ldots, \mathbf{x}_k = \{x_{k,1}, x_{k,2}, \ldots\}$ infinite sets of variables

- $\mu = (\mu^1, \ldots, \mu^k) \in \mathcal{P}^{\{1,\ldots,k\}} \rightsquigarrow \mathbb{H}_\mu(z,w) \in \mathbb{Q}(z,w)$ by the property that they satisfy the *k-point genus g Cauchy formula*:

$$\sum_{\lambda\in\mathcal{P}} \left(\prod_{i=1}^{k} \tilde{H}_\lambda(\mathbf{x}_i; z^2, w^2) \right) \mathcal{H}_\lambda(z,w) =$$

$$\mathrm{Exp}\left(\sum_{\mu\in\mathcal{P}^{\{1,\ldots,k\}}} \frac{\mathbb{H}_\mu(z,w)}{(z^2-1)(1-w^2)} m_{\mu^1}(\mathbf{x}_1)\ldots m_{\mu^k}(\mathbf{x}_k) \right),$$

where

$$\mathcal{H}_\lambda(z,w) := \prod \frac{(z^{2a+1}-w^{2l+1})^{2g}}{(z^{2a+2}-w^{2l})(z^{2a}-w^{2l+2})}$$

Main conjecture

Conjecture 2 (Hausel-Letellier-Villegas, 2007)**.** *The compactly supported mixed Hodge polynomial of* $\mathcal{M}_\mu = \mathcal{M}(\Sigma_\mu, \mathrm{GL}_n)$ *is given by*

$$H_c(\mathcal{M}_\mu; q, t) = (t\sqrt{q})^{d_\mu} \mathbb{H}_\mu \left(-\frac{1}{\sqrt{q}}, t\sqrt{q} \right),$$

where $d_\mu = \dim \mathcal{M}_\mu$.

Example. When $g = 0, k = 2$, and $\mu = (\mu^1, \mu^2)$, then $\mathcal{M}_\mu$ is empty, unless $\mu = ((1),(1))$, when it is a point. Then Conjecture 2 $\rightsquigarrow$ Cauchy formula for Macdonald polynomials.

Main result

Theorem 2 (Hausel-Letellier-Villegas, 2007). *Via the character table of* $\mathrm{GL}_n(\mathbb{F}_q)$ *by (Green 1955) the* E*-polynomial of the character variety* $\mathcal{M}_\mu = \mathcal{M}(\Sigma_\mu, \mathrm{GL}_n)$ *is*

$$E(\mathcal{M}_\mu; q) = H_c(\mathcal{M}_\mu; q, -1) = q^{d_\mu/2}\mathbb{H}_\mu\left(\frac{1}{\sqrt{q}}, \sqrt{q}\right) =$$

$$= (q-1) \sum_{\chi \in \mathrm{Irr}(\mathrm{GL_n}(\mathbb{F}_\mathrm{q}))} \frac{|\mathrm{GL}_n(\mathbb{F}_q)|^{2g-2}}{\dim(\chi)^{2g-2}} \prod_i^k \frac{\chi(\tilde{C}_i)}{\chi(1)} |\tilde{C}_i|$$

Via character table of $\mathfrak{gl}_n(\mathbb{F}_q)$ *by (Letellier 2005) the* E*-polynomial of the quiver variety* $\mathcal{Q}_\mu = \mathcal{M}(\Sigma_\mu, \mathfrak{gl}_n)$ *for indivisible* μ *is:*

$$E(\mathcal{Q}_\mu; q) = q^{d_\mu/2}\mathbb{H}_\mu(0, \sqrt{q}) =$$

$$= \frac{|\mathfrak{gl}_n(\mathbb{F}_q)|^{g-1}}{|\mathrm{PGL}_n(\mathbb{F}_q)|} \sum_{x \in \mathfrak{gl}_n(\mathbb{F}_q)^*} |C_{\mathfrak{gl}_n}(x)|^g \prod_{i=1}^k \mathcal{F}(1_{\mathcal{C}_i})(x),$$

Main Problem

Is there a TQFT and a "group" $\mathcal{GL}_n$ such that

$$Z_?(\Sigma_\mu, \mathcal{GL}_n) = H_c(\mathcal{M}_\mu; q, t) \quad ?$$

The "group" $\mathcal{GL}_n$ should be a common deformation of GL_n and $\mathfrak{gl}_n$ in the sense that

$$Z_?(\Sigma_\mu, \mathcal{GL}_n)|_{t=-1} = Z_{CS}(\Sigma_\mu, \mathrm{GL}_n(\mathbb{F}_q))$$

and the pure part

$$PZ_?(\Sigma_\mu, \mathcal{GL}_n) = Z_{CS?}(\Sigma_\mu, \mathfrak{gl}_n(\mathbb{F}_q))$$

Example. When $n = 3$, all $\tilde{\mathcal{C}}_i$ are regular semisimple, i.e. all $\mu^i = (1,1,1)$, main Conjecture says:

$$H(\mathcal{M}_\mu, q, t) =$$

$$\frac{\left(\left(qt^2+1\right)\left(q^2t^4+qt^2+1\right)\right)^k}{\left(q^3t^6-1\right)\left(q^3t^4-1\right)\left(q^2t^4-1\right)\left(q^2t^2-1\right)}$$

$$-\frac{\left(3\,q^2t^4\left(qt^2+1\right)\right)^k}{q^4t^8\left(q^2t^4-1\right)\left(q^2t^2-1\right)\left(qt^2-1\right)(q-1)}$$

$$+1/3\,\frac{6^k\left(qt^2\right)^{3\,k}}{q^6t^{12}\left(qt^2-1\right)^2(q-1)^2}$$

$$+\frac{\left(q^2t^4\left(2\,q^2t^2+qt^2+q+2\right)\right)^k}{q^4t^8\left(q^3t^4-1\right)\left(q^3t^2-1\right)\left(qt^2-1\right)(q-1)}$$

$$+\frac{\left(q^3t^6(q+1)\left(q^2+q+1\right)\right)^k}{q^6t^{12}\left(q^3t^2-1\right)\left(q^3-1\right)\left(q^2t^2-1\right)\left(q^2-1\right)}$$

$$-\frac{\left(3\,q^3t^6(q+1)\right)^k}{q^6t^{12}\left(q^2t^2-1\right)\left(q^2-1\right)\left(qt^2-1\right)(q-1)},$$