Mirror symmetry in the character table of $\mathrm{SL}_n(\mathbb{F}_q)$

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Quote from (Atiyah 2004)

”I would like to make a rash prediction that ideas from physics will have a big impact on number theory as the ideas flow across mathematics - on one extreme number theory, on the other physics, and in the middle geometry: the wind is blowing, and it will eventually reach to the farthest extremities of number theory and give us a new point of view.”

Cartoon I

Maxwell's equations:
$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}$$
electro-magnetic duality: $(\mathbf{E}, \mathbf{B}) \leftrightarrow (\mathbf{B}, -\mathbf{E})$

$\Downarrow$

S-duality for $N = 4$ SUSY Yang–Mills gauge groups G and G^L
(Montonen–Olive 1977)

$\Downarrow$

T-duality between σ-models with target $\mathcal{M}_{Hitchin} \cong \mathcal{M}_{\mathrm{Dol}} \cong \mathcal{M}_{\mathrm{DR}} \cong \mathcal{M}_{\mathrm{B}}$
(Bershadsky–Johanssen–Sadov–Vafa 1995)

$\Downarrow$

Cartoon II

(Strominger–Yau–Zaslow 1996) mirror symmetry for

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{Dol}}(\mathrm{G}) & & \mathcal{M}_{\mathrm{Dol}}(\mathrm{G}^L) \\ & {\scriptstyle \chi_{\mathrm{G}}} \searrow \quad \swarrow {\scriptstyle \chi_{\mathrm{G}^L}} & \\ & \mathcal{A} & \end{array}$$

(Hausel–Thaddeus 2003; Donagi–Pantev 2012)

$\Downarrow$

Homological mirror symmetry = Geometric Langlands

$$\mathcal{D}^b(Coh(\mathcal{M}_{\mathrm{DR}}(\mathrm{G}))) \sim \mathcal{D}^b(Fuk(\mathcal{M}_{\mathrm{DR}}(\mathrm{G}^L)))$$

(Kontsevich 1994; Laumon 1987)

(Beilinson–Drinfeld 1995; Kapustin–Witten 2007)

$\Downarrow$

semi-classical limit $\rightsquigarrow$ fiberwise Fourier-Mukai transform

$$\mathcal{D}^b(Coh(\mathcal{M}_{\mathrm{Dol}}(\mathrm{G}))) \sim \mathcal{D}^b(Coh(\mathcal{M}_{\mathrm{Dol}}(\mathrm{G}^L)))$$

(Arinkin 2002, Donagi–Pantev 2009)

$\Downarrow$

Cartoon III

cohomological shadow of fiberwise Fourier–Mukai transform

$$H^*_{str}(\mathcal{M}_{\mathrm{Dol}}(\mathrm{G})) \cong H^*_{str}(\mathcal{M}_{\mathrm{Dol}}(\mathrm{G}^L))$$

(Hausel–Thaddeus 2003; Ngô 2010)

$$\Downarrow$$

$P = W$ conjecture of (de Cataldo–Hausel–Migliorini, 2012) $\rightsquigarrow$

$$H^*_{str}(\mathcal{M}_{\mathrm{B}}(\mathrm{G})) \cong H^*_{str}(\mathcal{M}_{\mathrm{B}}(\mathrm{G}^L))$$

(Hausel–Villegas 2004)

$$\Downarrow$$

Frobenius formula $\rightsquigarrow$ identities in $\mathrm{Irr}(\mathrm{G}(\mathbb{F}_q))$ and $\mathrm{Irr}(\mathrm{G}^L(\mathbb{F}_q)) \rightsquigarrow$

$$\#_{str}(\mathcal{M}_{\mathrm{B}}(\mathrm{G})(\mathbb{F}_q)) = \#_{str}(\mathcal{M}_{\mathrm{B}}(\mathrm{G}^L)(\mathbb{F}_q))$$

(Hausel–Mereb–Villegas 2012)

SL_n & PGL_n Higgs moduli spaces

- C smooth projective curve; $\mathrm{G} = \mathrm{SL}_n$ $\mathrm{G}^L = \mathrm{PGL}_n$; $\Lambda \in \mathrm{Pic}^d(C)$
- Λ-twisted SL_n Higgs bundle (E, ϕ)
  $\phi \in H^0(C; \mathrm{End}_0(E) \otimes K)$ and $\det(E) = \Lambda$
- $\mathcal{M}_{\mathrm{Dol}}(\mathrm{SL}_n)$ moduli of semi-stable Λ-twisted SL_n Higgs bundles
- $(n, d) = 1 \Rightarrow \mathcal{M}_{\mathrm{Dol}}(\mathrm{SL}_n)$ non-singular quasi-projective
- $\Gamma := Jac(C)[n] \cong (\mu_n)^{2g}$ acts by: $(E, \phi) \mapsto (E \otimes L, \phi)$
- PGL_n Higgs moduli space is the orbifold:
  $\mathcal{M}_{\mathrm{Dol}}(\mathrm{PGL}_n) := \mathcal{M}_{\mathrm{Dol}}(\mathrm{SL}_n)/\Gamma$

Hitchin map and SYZ

- Hitchin map: $\chi_{\mathrm{SL}_n} : \begin{array}{ccc} \mathcal{M}_{\mathrm{Dol}}(\mathrm{SL}_n) & \to & \mathcal{A} := \bigoplus_{i=2}^{n} H^0(C; K^i) \\ (E,\phi) & \mapsto & \mathrm{charpol}(\phi) \end{array}$

Theorem (Hitchin 1987,1989; Nitsure 1991; Faltings 1993)

χ_{SL_n} *is proper completely integrable system.*

Theorem (Hausel–Thaddeus 2003)

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{Dol}}(\mathrm{SL}_n) & & \mathcal{M}_{\mathrm{Dol}}(\mathrm{PGL}_n) \\ & {\scriptstyle \chi_{\mathrm{SL}_n}} \searrow \quad \swarrow {\scriptstyle \chi_{\mathrm{PGL}_n}} & \\ & \mathcal{A} & \end{array}$$

satisfies (Strominger-Yau-Zaslow 1996) for mirror symmetry.

Conjecture (Hausel-Thaddeus 2003)

$$
\begin{array}{ccc}
H^*_{str}(\mathcal{M}_{\mathrm{Dol}}(\mathrm{SL}_n);\mathbb{Q}) & \cong & H^*_{str,B}(\mathcal{M}_{\mathrm{Dol}}(\mathrm{PGL}_n);\mathbb{Q}) \\
\cong & & \cong \\
\bigoplus_{\kappa\in\hat{\Gamma}} H^*(\mathcal{M}_{\mathrm{Dol}}(\mathrm{SL}_n);\mathbb{Q})_\kappa & \cong & \bigoplus_{\gamma\in\Gamma} H^{*-\mathrm{codim}\mathcal{M}^\gamma}(\mathcal{M}_{\mathrm{Dol}}(\mathrm{SL}_n)^\gamma/\Gamma, L^B_\gamma)
\end{array}
$$

Results:

- $n = 2, 3$ by (Hausel, Thaddeus 2003)
- for all n in the middle degree $* = \dim(\mathcal{M}_{\mathrm{Dol}}(\mathrm{SL}_n))$ by (Garcia-Prada–Heinloth–Schmidt, 2010) using (Laumon, 1987)
- for all n up to degree $* < \min_{\gamma\in\Gamma^*} \mathrm{codim}\mathcal{M}^\gamma$ by (Hausel–Pauly 2012) using symmetries of Hitchin fibers (Ngô, 2006)

Perverse filtration

- for any proper map $f : X \to Y$ (de Cataldo-Migliorini 2005) introduce *perverse filtration* $\subset P_i \subset P_{i+1} \subset \dots P_k \cong H^k(X)$ from the study of the Beilinson-Bernstein-Deligne-Gabber decomposition theorem for $f_!(\mathbb{Q}_X)$ into perverse sheaves
- extending the geometric fundamental lemma of (Ngô, 2010) from $\mathcal{A}_{ell}$ over $\mathcal{A} \rightsquigarrow$

Conjecture (Refined topological mirror test)

Under the Weil pairing $\Gamma^* \cong \Gamma$ *given by* $\kappa \leftrightarrow \gamma$

$$Gr^P_k H^*(\mathcal{M}_{\mathrm{Dol}}(\mathrm{SL}_n))_\kappa \cong Gr^P_{k-\frac{F(\gamma)}{2}} H^{*-F(\gamma)}(\mathcal{M}_{\mathrm{Dol}}(\mathrm{SL}_n)^\gamma/\Gamma, L^B_\gamma)$$

- the SL_n-character variety:

$$\mathcal{M}_{\mathrm{B}}(\mathrm{SL}_n) := \{(A_i, B_i)_{i=1..g} \in \mathrm{SL}_n^{2g} \mid [A_1, B_1] \ldots [A_g, B_g] = \zeta_n^d I_n\} // \mathrm{PGL}_n$$

non-singular, affine

- for PGL_n note that $\Gamma \cong (\mu_n)^{2g}$ acts on $\mathcal{M}_{\mathrm{B}}(\mathrm{SL}_n)$

$\mathcal{M}_{\mathrm{B}}(\mathrm{PGL}_n) := \mathcal{M}_{\mathrm{B}}(\mathrm{SL}_n)/\Gamma$

Theorem (Non-Abelian Hodge Theorem; Simpson, Corlette)

$$\mathcal{M}_{\mathrm{Dol}} \overset{\mathit{diff}}{\cong} \mathcal{M}_{\mathrm{B}}$$

$P = W$ and Betti Mirror Symmetry

Conjecture ("P=W", de Cataldo-Hausel-Migliorini 2012)

$P_k(\mathcal{M}_{\mathrm{Dol}}) \cong W_{2k}(\mathcal{M}_{\mathrm{B}})$ *under the isomorphism* $H^*(\mathcal{M}_{\mathrm{Dol}}) \cong H^*(\mathcal{M}_{\mathrm{B}})$ *from non-Abelian Hodge theory.*

Conjecture (Hausel–Villegas 2004)

Under the Weil pairing $\Gamma^* \cong \Gamma$ *given by* $\kappa \leftrightarrow \gamma$

$$Gr^W_k H^*(\mathcal{M}_{\mathrm{B}}(\mathrm{SL}_n))_\kappa \cong Gr^W_{2k-F(\gamma)} H^{*-F(\gamma)}(\mathcal{M}_{\mathrm{B}}(\mathrm{SL}_n)^\gamma/\Gamma, L^B_\gamma)$$

in particular

$$\begin{array}{ccc} E_\kappa(\mathcal{M}_{\mathrm{B}}(\mathrm{SL}_n)) & := & \sum_{i,k} |Gr^W_k H^i(\mathcal{M}_{\mathrm{B}}(\mathrm{SL}_n))_\kappa| q^k (-1)^i \\ \| & & \| \end{array}$$

$$E(\mathcal{M}_{\mathrm{B}}(\mathrm{SL}_n)^\gamma/\Gamma, L^B_\gamma) q^{F(\gamma)} = \sum_{k,i} |Gr^W_{2k} H^i(\mathcal{M}_{\mathrm{B}}(\mathrm{SL}_n)^\gamma/\Gamma, L^B_\gamma)| q^{k+F(\gamma)} (-1)^i$$

Arithmetic harmonic analysis

Theorem (Katz, 2008)

When $X/\mathbb{Z}$ has polynomial-count

$$\#(X(\mathbb{F}_q)) = E(X/\mathbb{C}, q) = \sum_{i,k} |Gr_k^W H^i(X/\mathbb{C})| q^k (-1)^i$$

Theorem (Frobenius, 1896)

For any finite group G:

$$\#\left\{a_1, b_1, \ldots, a_g, b_g \in G | \prod [a_i, b_i] = z\right\} = \sum_{\chi \in \mathrm{Irr}(G)} \frac{|G|^{2g-1}}{\chi(1)^{2g-1}} \chi(z)$$

- for $G = \mathrm{GL}_n(\mathbb{F}_q)$ (Hausel–Letellier–Villegas, 2006–2012) have evaluated this and more general character formulas $\rightsquigarrow$ complete conjectures about the mixed Hodge polynomials $\sum_{i,k} |Gr_k^W H^i(\mathcal{M}_{\mathrm{B}}(\mathrm{GL}_n))| q^k t^i$ using Macdonald polynomials
- for $G = \mathrm{SL}_n(\mathbb{F}_q)$ and $z = \exp(\frac{2\pi i d}{n})$ the character formula was evaluated in (Mereb, 2010)

Character formulas for refined Betti mirror symmetry

- $\kappa \in \hat{\Gamma}$ and $\epsilon \in \hat{\mathbb{F}}_q^\times$ such that $ord(\kappa) = ord(\epsilon) = k$

$$E_\kappa(\mathcal{M}_{\mathrm{B}}(\mathrm{SL}_n)) = \sum_{\substack{\theta\in\mathrm{Irr}(\mathrm{SL}_n(\mathbb{F}_q))\\ k \| \theta|}} |\theta|^{-2g}\left(\frac{|\mathrm{SL}_n(\mathbb{F}_q)|}{\theta(1)}\right)^{2g-2}\frac{\theta(\zeta_n 1)}{\theta(1)}$$

$$= \frac{1}{(q-1)^{2g-1}}\sum_{\substack{\chi\in\mathrm{Irr}(\mathrm{GL}_n(\mathbb{F}_q))\\ \chi=\epsilon\chi}}\left(\frac{|\mathrm{GL}_n(\mathbb{F}_q)|}{\chi(1)}\right)^{2g-2}\frac{\chi(\zeta_n 1)}{\chi(1)}$$

- when $k = ord(\gamma)$

$$E(\mathcal{M}_{\mathrm{B}}(\mathrm{SL}_n)^\gamma/\Gamma, L^B_\gamma)q^{F(\gamma)} =$$

$$\sum_{s|k}\frac{\mu(s)q^{n^2\frac{k-1}{k}(g-1)}}{k(q-1)^{2g-1}}\sum_{\substack{\chi\in\mathrm{Irr}\left(\mathrm{GL}_{n/k}(\mathbb{F}_{q^s})\right)\\ \chi=\chi^{\mathrm{Frob}_q}}}\left(\frac{|\mathrm{GL}_{n/k}(\mathbb{F}_{q^s})|}{\chi(1)}\right)^{(2g-2)k/s}\left(\frac{\chi(\zeta_n 1)}{\chi(1)}\right)^{k/s}$$

Main result

Theorem (Hausel–Mereb–Villegas 2012)

When $\kappa \in \hat{\Gamma}$ *and* $\gamma \in \Gamma$ *such that* $ord(\kappa) = ord(\gamma) = k$

$$E(\mathcal{M}_{\mathrm{B}}(\mathrm{SL}_n)^{\gamma}/\Gamma, L_{\gamma}^{B})q^{F(\gamma)} =$$

$$= \frac{q^{n^2(g-1)}}{k(q-1)^{2g-2}} \sum_{u|k^{\infty}} \frac{1}{u} \sum_{s|k} \mu(s) \frac{q^{-\frac{(g-1)n^2}{uk}}}{(q^{us}-1)^2} E\left(\mathcal{M}_{\mathrm{B}}^{(\frac{k}{s})}(\mathrm{GL}_{\frac{n}{ku}}); q^{us}\right) =$$

$$= E_{\kappa}(\mathcal{M}_{\mathrm{B}}(\mathrm{SL}_n)).$$

- one can compute from this $\chi(\mathcal{M}_{\mathrm{B}}(\mathrm{SL}_n)) = \mu(n)n^{4g-3}$ matching (Mereb 2010)
- this has a natural t-deformation giving a conjecture for the mixed Hodge polynomial of $\mathcal{M}_{\mathrm{B}}(\mathrm{SL}_n)$ and in turn the mixed Hodge polynomial of $\mathcal{M}_{\mathrm{Dol}}(\mathrm{SL}_n)$

Example for Betti mirror symmetry for $n = 2$

- for $k = ord(\gamma) = 2$:

$$E(\mathcal{M}_{\mathrm{B}}(\mathrm{SL}_2)^\gamma/\Gamma, L^B_\gamma)q^{F(\gamma)} =$$

$$\sum_{s|2} \frac{\mu(s)q^{2(g-1)}}{2(q-1)^{2g-1}} \sum_{\substack{\chi\in\mathrm{Irr}\left(\mathrm{GL}_1(\mathbb{F}_{q^s})\right)\\ \chi=\chi^{\mathrm{Frob}_q}}} \left(\frac{|\mathrm{GL}_1(\mathbb{F}_{q^s})|}{\chi(1)}\right)^{(2g-2)2/s} \left(\frac{\chi(\zeta_2 1)}{\chi(1)}\right)^{2/s}$$

$$= \left(\frac{(q-1)^{2g-2}-(q+1)^{2g-2}}{2}\right) q^{2g-2}$$

- for $k = ord(\kappa) = 2$:

$$E_\kappa(\mathcal{M}_{\mathrm{B}}(\mathrm{SL}_2)) = \sum_{\substack{\theta\in\mathrm{Irr}(\mathrm{SL}_2(\mathbb{F}_q))\\ 2||\theta|}} |\theta|^{-2g} \left(\frac{|\mathrm{SL}_2(\mathbb{F}_q)|}{\theta(1)}\right)^{2g-2} \frac{\theta(\zeta_2 1)}{\theta(1)}$$

$$= \left(\frac{(q-1)^{2g-2}-(q+1)^{2g-2}}{2}\right) q^{2g-2}$$

- character tables for $\mathrm{GL}_2(\mathbb{F}_q)$ and $\mathrm{SL}_2(\mathbb{F}_q)$ known to (Jordan 1907) and (Schur 1907)

Schur's character table of $\mathrm{SL}_2(\mathbb{F}_q)$

2. Die Gruppe $\mathfrak{L}_{p^n}$, die durch die ganzen linearen Substitutionen

$$\xi_1 = \alpha\eta_1 + \beta\eta_2, \quad \xi_2 = \gamma\eta_1 + \delta\eta_2$$

gebildet wird, deren Determinante gleich 1 ist. — Die Ordnung der Gruppe

Die $s+4$ Charaktere von $\mathfrak{L}_s$ lassen sich in folgender Tabelle zusammenfassen:

	1	1	$\frac{s-3}{2}$	$\frac{s-1}{2}$	2	2
$\chi(E)$	1	s	$s+1$	$s-1$	$\frac{1}{2}(s+1)$	$\frac{1}{2}(s-1)$
$\chi(F)$	1	s	$(-1)^{\alpha}(s+1)$	$(-1)^{\beta}(s-1)$	$\frac{\varepsilon}{2}(s+1)$	$-\frac{\varepsilon}{2}(s-1)$
$\chi(P)$	1	0	1	-1	$\frac{1}{2}(1 \pm \sqrt{\varepsilon s})$	$\frac{1}{2}(-1 \pm \sqrt{\varepsilon s})$
$\chi(Q)$	1	0	1	-1	$\frac{1}{2}(1 \mp \sqrt{\varepsilon s})$	$\frac{1}{2}(-1 \mp \sqrt{\varepsilon s})$
$\chi(A^a)$	1	1	$\varrho^{\alpha a} + \varrho^{-\alpha a}$	0	$(-1)^a$	0
$\chi(B^b)$	1	-1	0	$-(\sigma^{\beta b} + \sigma^{-\beta b})$	0	$-(-1)^b$.

*) Vgl. *Dickson*, Linear Groups, Cap. XII.

Betti mirror symmetry for general reductive group G?

- let G be complex reductive
- (Hausel–Letellier–Villegas 2012) $\rightsquigarrow$ definition for $H^*_{str}(T^*V////\mathrm{G})$ for the stringy cohomology of symplectic quotient attached to any representation $\mathrm{G} \to \mathrm{GL}(V)$
- $\mu: \begin{array}{ccc} \mathrm{G}^{2g} & \to & \mathrm{G} \\ (A_i, B_i)_1^g & \mapsto & \Pi_i[A_i, B_i] \end{array}$ group valued moment map equivariant
- usual quotient $\mathcal{M}_{\mathrm{B}}(\mathrm{G}) := \mu^{-1}(1)//\mathrm{G}$
- $\mathcal{C} \subset \mathrm{G}$ *generic* regular semisimple conjugacy class
- $\tilde{\mathcal{M}}_{\mathrm{B}}(\mathrm{G}) := \mu^{-1}(\mathcal{C})//\mathrm{G}$ orbifold and the Weyl group W of G acts on $H^*_{str}(\tilde{\mathcal{M}}_{\mathrm{B}}(\mathrm{G}))$
- $H^*_{str}(\mathcal{M}_{\mathrm{B}}(\mathrm{G})) := H^*_{str}(\tilde{\mathcal{M}}_{\mathrm{B}}(\mathrm{G}))_\epsilon$ where ϵ is the sign representation of W on $H^{mid}(\mathcal{C})$

Conjecture

$$H^*_{str}(\mathcal{M}_{\mathrm{B}}(\mathrm{G})) \cong H^*_{str}(\mathcal{M}_{\mathrm{B}}(\mathrm{G}^L))$$