Mirror symmetry Langlands duality and the Hitchin system III

Tamás Hausel

Royal Society URF at University of Oxford
http://www.maths.ox.ac.uk/∼hausel/talks.html

October 2010
Simons lectures
SUNY Stony Brook

Problems with topological mirror symmetry

- C smooth projective curve, $(n,d)=1$
- $\check{\mathcal{M}}, \hat{\mathcal{M}}$ moduli of stable SL_n,PGL_n-Higgs bundles on C
- $\check{\mathcal{M}}_{\mathrm{B}}$, $\hat{\mathcal{M}}_{\mathrm{B}}$ twisted character variety of $\pi_1(C) \to \mathrm{SL}_n, \mathrm{PGL}_n$

Conjecture (Hausel–Thaddeus, 2003)

$$E(\check{\mathcal{M}}; x, y) = E_{\mathrm{st}}^{\hat{B}}(\hat{\mathcal{M}}; x, y)$$

Conjecture (Hausel–Villegas 2004)

$$E(\check{\mathcal{M}}_{\mathrm{B}}; x, y) = E_{\mathrm{st}}^{\hat{B}}(\hat{\mathcal{M}}_{\mathrm{B}}; x, y).$$

1. Why the same Hodge numbers, why not mirrored ones?
2. Why two different topolocial mirror symmetry conjectures?
3. Connection to geometric/classical Langlands?

- $g = n = 2 \rightsquigarrow E(\check{\mathcal{M}}_{\mathrm{B}}; x, y) = q^6 - 2q^4 - 30q^3 - 2q^2 + 1$
- $E(\mathring{\hat{\mathcal{M}}}_{\mathrm{B}}; 1/x, 1/y) = (xy)^{d_n} E(\mathring{\hat{\mathcal{M}}}_{\mathrm{B}}; x, y)$
- Alvis-Curtis duality

$$q^k \chi(1)(1/q) = \chi'(1)(q) \text{ for dual pair } \chi, \chi' \in Irr(\mathrm{G}(\mathbb{F}_q))$$

Conjecture (Hausel–Villegas, 2008)

$$L^l : \begin{array}{ccc} Gr^W_{d_n-2l}(H^{i-l}(\mathcal{M}_{\mathrm{B}})) & \overset{\cong}{\to} & Gr^W_{d_n+2l} H^{i+l}(\mathcal{M}_{\mathrm{B}}) \\ x & \mapsto & x \cup \alpha^l \end{array},$$

where $\alpha \in W_4 H^2(\mathcal{M}_{\mathrm{B}})$

- for $n = 2$ theorem of (Hausel–Villegas, 2008)

Perverse filtration

- Perverse filtration: $P_0 \subset \cdots \subset P_i \subset \cdots \subset P_k(X) \cong H^k(X)$ for $f : X \to Y$ proper from BBDG decomposition theorem
- X smooth Y affine (de Cataldo-Migliorini, 2008): $Y_0 \subset \cdots \subset Y_i \subset \cdots \subset Y_l = Y$ s.t. Y_i generic with $\dim(Y_i) = i$

$$P_{k-i-1}H^k(X) = \mathrm{ker}(\mathrm{H^k(X)} \to \mathrm{H^k(f^{-1}(Y_i))})$$

- Relative Hard Lefschetz Theorem:

$$\begin{array}{rccc} L^l : & Gr^P_{d-l}(H^*(X)) & \xrightarrow{\cong} & Gr^P_{d+l}H^{*+2l}(X) \\ & x & \mapsto & x \cup \alpha^l \end{array}$$

$\alpha \in H^2(X)$ relative ample class

- recall Hitchin map $\chi : \begin{array}{ccc} \mathcal{M} & \rightarrow & \mathcal{A} \\ (E, \phi) & \mapsto & \mathrm{charpol}(\phi) \end{array}$ is proper

  $\rightsquigarrow$ perverse filtration on $H^*(\mathcal{M})$

Conjecture ("P=W", de Cataldo-Hausel-Migliorini 2010)

$P_k(\mathcal{M}) \cong W_{2k}(\mathcal{M}_{\mathrm{B}})$ *under the isomorphism* $H^*(\mathcal{M}) \overset{NAH}{\cong} H^*(\mathcal{M}_{\mathrm{B}})$.

Theorem (de Cataldo-Hausel-Migliorini 2010)

$P = W$ *for* $G = \mathrm{GL}_2, \mathrm{PGL}_2, \mathrm{SL}_2$.

- proof mirrors and refines special cases of (Ngô 2008)

Unified topological mirror symmetry

- Define $PE(\mathcal{M}; x, y, q) := \sum_{k,i,j} h^{i,j}(Gr_k^P(H^{i+j}(\mathcal{M})))x^i y^j q^k$
- $PE(\mathcal{M}; -\frac{1}{x}, -\frac{1}{y}, 1)(xy)^d = E(\mathcal{M}; x, y)$
- Conjecture $P = W \Rightarrow PE(\mathcal{M}; -1, -1, q) = E(\mathcal{M}_{\mathrm{B}}; \sqrt{q}, \sqrt{q})$
- RHL $\rightsquigarrow$ $PE(\mathcal{M}; x, y, q) = (xyq)^d PE(\mathcal{M}; x, y; \frac{1}{qxy}) \rightsquigarrow$

Conjecture (Topological Mirror test, TMS)

$$PE\left(\check{\mathcal{M}}^d; x, y, q\right) = (xyq)^{d_n} PE_{\mathrm{st}}^{\hat{B}^d}\left(\hat{\mathcal{M}}^d; x, y, \frac{1}{qxy}\right)$$

Example for RHL

- let $\check{\mathcal{M}}$ be moduli space of SL_2 parabolic Higgs bundles on elliptic curve E with one parabolic point
- $\check{\chi} : \check{\mathcal{M}} \to \mathcal{A} \cong \mathbb{A}^1$ is elliptic fibration, with singular fibre $\check{\chi}^{-1}(0)$ of type $\hat{D}_4$
- $P_1(H^2(\check{\mathcal{M}})) = \ker(H^2(\check{\mathcal{M}}) \to H^2(\chi^{-1}(pt)) \cong \mathrm{im}(H^2_{cpt}(\check{\mathcal{M}}) \to H^2(\check{\mathcal{M}}))$ has dimension 4
- $E(\check{\mathcal{M}}; x, y) = 5xy + (xy)^2$ non-symmetric
- $PE(\check{\mathcal{M}}; x, y, q) = 1 + 4xyq + xyq^2$ symmetric
- $E(\check{\mathcal{M}}_{\mathrm{B}}; q) = PE(\check{\mathcal{M}}; -1, -1, q) = 1 + 4q + q^2$
- RHL $\rightsquigarrow$ $PE(\check{\mathcal{M}}; x, y, q) = (xyq)^2 PE(\check{\mathcal{M}}; x, y; \frac{1}{qxy})$

Example for TMS

- $\mathbb{Z}_2$ acts on E and $\mathbb{C}$ as additive inverse $x \mapsto -x$
- $\check{\mathcal{M}} \to E \times \mathbb{C}/\mathbb{Z}_2$ blowing up; $\chi : \check{\mathcal{M}} \to \mathbb{C}/\mathbb{Z}_2 \cong \mathbb{C}$ is elliptic fibration with $\hat{D}_4$ singular fiber over 0
- $\Gamma = E[2] \cong \mathbb{Z}_2^2$ acts on $\check{\mathcal{M}}$ by multiplying on E
- $\hat{\mathcal{M}}$ the PGL_2 moduli space is $\check{\mathcal{M}}/\Gamma$ an orbifold, elliptic fibration over $\mathbb{C}$ with A_1 singular fiber
  with three $\mathbb{C}^2/\mathbb{Z}_2$-orbifold points on one of the components
- blowing up the three orbifold singularities is crepant gives $\check{\mathcal{M}}$
- original topological mirror test:
  $E_{st}(\hat{\mathcal{M}}; x, y) \overset{Kontsevich}{=} E(\check{\mathcal{M}}; x, y)$
- enhances to $PE_{st}(\hat{\mathcal{M}}; x, y, q) = PE(\check{\mathcal{M}}; x, y, q)$
- RHL $\rightsquigarrow \mathrm{PE}_{st}(\hat{\mathcal{M}}; x, y, q) = (xyq)^2 PE(\check{\mathcal{M}}; x, y, \frac{1}{xyq})$

From S-duality to the Fundamental Lemma

- Maxwell's (1861) equations in vacuum:

$$\begin{array}{cc} \nabla \cdot \mathbf{E} = 0 & \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} & \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} \end{array}$$

- electro magnetic duality: $(\mathbf{E}, \mathbf{B}) \leftrightarrow (\mathbf{B}, -\mathbf{E}) \rightsquigarrow$ S-duality
- (Kapustin–Witten, 2006) S-duality reduces to HMS between $\check{\mathcal{M}}$ and $\hat{\mathcal{M}} \leftrightarrow$ Geometric Langlands
- semi-classical limit: fiberwise Fourier-Mukai transform
- (de Cataldo-Hausel-Migliorini 2010)

  $FM : H^*_p(\check{\mathcal{M}}_{\mathrm{reg}}) \cong H^{*+2d_n-2p}_{\mathrm{st},2d_n-p}(\hat{\mathcal{M}}_{\mathrm{reg}})$
- $\rightsquigarrow$ topological mirror symmetry conjecture when n is prime
- $\rightsquigarrow$ Ngo's formula for SL_n:
  $H^*_p(\check{\mathcal{M}}_a)_\kappa \cong H^{*-2F(w(\kappa))}_{p-2F(w(\kappa))}(\check{\mathcal{M}}^{w(\kappa)}_a/\Gamma)$

  $\rightsquigarrow$ Fundamental Lemma in the classical Langlands program

Quote from (Atiyah 2004)

"I would like to make a rash prediction that ideas from physics will have a big impact on number theory as the ideas flow across mathematics - on one extreme number theory, on the other physics, and in the middle geometry: the wind is blowing, and it will eventually reach to the farthest extremities of number theory and give us a new point of view."