Mirror symmetry Langlands duality and the Hitchin system II

Tamás Hausel

Royal Society URF at University of Oxford
http://www.maths.ox.ac.uk/∼hausel/talks.html

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Frobenius on group characters

- FROBENIUS, F.G.: Über Gruppencharaktere (1896):

  " I shall develop the concept [of character for arbitrary finite groups] here in the belief that through its introduction, group theory will be substantially enriched. "

- After proving the orthogonality relations ($k = 2; g = 0$ below) Frobenius' first application was the $g = 0$ case of

Theorem (Frobenius 1896, Mednykh 1978,Freed-Quinn 1993,...)

Let $\mathcal{C}_1, \ldots, \mathcal{C}_k \subset \mathrm{G}$ be conjugacy classes in a finite group G then

$$\#\{c_i \in \mathcal{C}_i, a_j, b_j \in \mathrm{G} | c_1 c_2 \cdots c_k [a_1, b_1] \ldots [a_g, b_g] = 1\} =$$

$$= \sum_{\chi \in \mathrm{Irr}(\mathrm{G})} \frac{|\mathrm{G}|^{2g-1}}{\chi(1)^{2g-2}} \prod_{i=1}^{k} \frac{\chi(\mathcal{C}_i)|\mathcal{C}_i|}{\chi(1)}$$

- $\mathrm{G} = \mathrm{GL}_n(\mathbb{F}_q)$
- character table of $\mathrm{GL}_n(\mathbb{F}_q)$ was calculated by
  (Jordan, Schur, 1907) for $n = 2$
  (Steinberg, 1951) for $n = 3, 4$
  (Green, 1955) for all n
- (Hausel–Letellier–Villegas, 2008) calculated explicitly (using Macdonald polynomials)

$$\sum_{\chi \in \mathrm{Irr}(\mathrm{GL}_n(\mathbb{F}_q))} \frac{|\mathrm{G}|^{2g-1}}{\chi(1)^{2g-2}} \prod_{i=1}^{k} \frac{\chi(\mathcal{C}_i)|\mathcal{C}_i|}{\chi(1)} =$$

$$\#\{c_i \in \mathcal{C}_i, a_j, b_j \in \mathrm{G} | c_1 c_2 \cdots c_k [a_1, b_1] \ldots [a_g, b_g] = 1\},$$

where $\mathcal{C}_i \subset \mathrm{GL}_n(\mathbb{F}_q)$ are generic semisimple conjugacy classes

Character table of $\mathrm{GL}_2(\mathbb{F}_q)$

Table 1: characters of $\mathbf{GL}_2(\mathbb{F}_q)$
(note that $|\mathbf{GL}_2(\mathbb{F}_q)| = q(q-1)^2(q+1)$)

Classes	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ $a \in \mathbb{F}_q^\times$	$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ $a, b \in \mathbb{F}_q^\times$ $a \neq b$	$\begin{pmatrix} x & 0 \\ 0 & {}^F x \end{pmatrix}$ $x \in \mathbb{F}_{q^2}^\times$ $x \neq {}^F x$	$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ $a \in \mathbb{F}_q^\times$
Number of classes of this type	$q-1$	$\frac{(q-1)(q-2)}{2}$	$\frac{q(q-1)}{2}$	$q-1$
Cardinal of the class	1	$q(q+1)$	$q(q-1)$	q^2-1
$R_{\mathbf{T}}^{\mathbf{G}}(\alpha, \beta)$ $\alpha, \beta \in \mathrm{Irr}(\mathbb{F}_q^\times)$ $\alpha \neq \beta$	$(q+1)\alpha(a)\beta(a)$	$\alpha(a)\beta(b) + \alpha(b)\beta(a)$	0	$\alpha(a)\beta(a)$
$-R_{\mathbf{T}_s}^{\mathbf{G}}(\omega)$ $\omega \in \mathrm{Irr}(\mathbb{F}_{q^2}^\times)$ $\omega \neq \omega^q$	$(q-1)\omega(a)$	0	$-\omega(x) - \omega({}^F x)$	$-\omega(a)$
$\mathrm{Id}_{\mathbf{G}}.(\alpha \circ \det)$ $\alpha \in \mathrm{Irr}(\mathbb{F}_q^\times)$	$\alpha(a^2)$	$\alpha(ab)$	$\alpha(x.{}^F x)$	$\alpha(a^2)$
$\mathrm{St}_{\mathbf{G}}.(\alpha \circ \det)$ $\alpha \in \mathrm{Irr}(\mathbb{F}_q^\times)$	$q\alpha(a^2)$	$\alpha(ab)$	$-\alpha(x.{}^F x)$	0

Example $\mathrm{GL}_2(\mathbb{F}_q)$

- $\mathrm{G} = \mathrm{GL}_2(\mathbb{F}_q)$, $k = 1$ $\mathcal{C}_1 = \{-1\} \subset \mathrm{GL}_2(\mathbb{F}_q)$
- $$\frac{1}{|\mathrm{PGL}_2(\mathbb{F}_q)|(q-1)^{2g}} \#\{a_j, b_j \in \mathrm{G} | [a_1, b_1] \ldots [a_g, b_g] = -1\} =$$

$$\frac{1}{|\mathrm{PGL}_2(\mathbb{F}_q)|(q-1)^{2g}} \sum_{\chi \in \mathrm{Irr}(\mathrm{GL}_2(\mathbb{F}_q))} \frac{|\mathrm{G}|^{2g-1}}{\chi(1)^{2g-2}} \frac{\chi(-1)}{\chi(1)} =$$

$$(q^2-1)^{2g-2} + q^{2g-2}(q^2-1)^{2g-2} - \frac{1}{2} q^{2g-2}(q-1)^{2g-2} - \frac{1}{2} q^{2g-2}(q+1)^{2g-2}.$$

- e.g. $g = 0$ gives 0 when $g = 1$ it gives 1

Example $\mathrm{SL}_n(\mathbb{F}_q)$

- character table for $\mathrm{SL}_2(\mathbb{F}_q)$ by (Jordan 1907), (Schur 1907) … for $\mathrm{SL}_n(\mathbb{F}_q)$ (Bonnafé 2006), (Shoji 2006)
- for $\mathrm{G} = \mathrm{SL}_2(\mathbb{F}_q)$, $k = 1$, $\mathcal{C}_1 = \{-1\} \subset \mathrm{SL}_2(\mathbb{F}_q)$
-

$$\frac{1}{|\mathrm{PGL}_2(\mathbb{F}_q)|} \#\{a_j, b_j \in \mathrm{G} | [a_1, b_1] \ldots [a_g, b_g] = -1\} =$$

$$\frac{1}{|\mathrm{PGL}_2(\mathbb{F}_q)|} \sum_{\chi \in \mathrm{Irr}(\mathrm{SL}_2(\mathbb{F}_q))} \frac{|\mathrm{G}|^{2g-1}}{\chi(1)^{2g-2}} \frac{\chi(-1)}{\chi(1)} =$$

$$(q^2-1)^{2g-2} + q^{2g-2}(q^2-1)^{2g-2} - \frac{1}{2} q^{2g-2}(q-1)^{2g-2} - \frac{1}{2} q^{2g-2}(q+1)^{2g-2} +$$

$$+ (2^{2g} - 1) q^{2g-2} \left(\frac{(q-1)^{2g-2} - (q+1)^{2g-2}}{2} \right).$$

Character table of $\mathrm{SL}_2(\mathbb{F}_q)$

Table 2: characters of $\mathbf{SL}_2(\mathbb{F}_q)$ for q odd
(note that $|\mathbf{SL}_2(\mathbb{F}_q)| = q(q-1)(q+1)$)

Classes	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ $a \in \{1,-1\}$	$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ $a \in \mathbb{F}_q^\times$ $a \neq \{1,-1\}$	$\begin{pmatrix} x & 0 \\ 0 & {}^F x \end{pmatrix}$ $x.{}^F x = 1$ $x \neq {}^F x$	$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ $a \in \{1,-1\}$, $b \in \{1, x\}$ with $x \in \mathbb{F}_q^\times - (\mathbb{F}_q^\times)^2$
Number of classes of this type	2	$(q-3)/2$	$(q-1)/2$	4
Cardinal of the class	1	$q(q+1)$	$q(q-1)$	$(q^2-1)/2$
$R_{\mathbf{T}}^{\mathbf{G}}(\alpha)$ $\alpha \in \mathrm{Irr}(\mathbb{F}_q^\times)$ $\alpha^2 \neq \mathrm{Id}$	$(q+1)\alpha(a)$	$\alpha(a) + \alpha(\frac{1}{a})$	0	$\alpha(a)$
$\chi_{\alpha_0}^{\varepsilon}$ $\varepsilon \in \{1,-1\}$	$\frac{q+1}{2}\alpha_0(a)$	$\alpha_0(a)$	0	$\frac{\alpha_0(a)}{2}(1-\varepsilon\alpha_0(ab)\sqrt{\alpha_0(-1)q})$
$-R_{\mathbf{T}_s}^{\mathbf{G}}(\omega)$ $\omega \in \mathrm{Irr}(\mu_{q+1})$ $\omega^2 \neq \mathrm{Id}$	$(q-1)\omega(a)$	0	$-\omega(x) - \omega({}^F x)$	$-\omega(a)$
$\chi_{\omega_0}^{\varepsilon}$ $\varepsilon \in \{1,-1\}$	$\frac{q-1}{2}\omega_0(a)$	0	$-\omega_0(x)$	$\frac{\omega_0(a)}{2}(-1+\varepsilon\alpha_0(ab)\sqrt{\alpha_0(-1)q})$
$\mathrm{Id}_{\mathbf{G}}$	1	1	1	1
$\mathrm{St}_{\mathbf{G}}$	q	1	-1	0

Character varieties for GL_n and SL_n

- fix integers $n > 1$ and d such that $(n, d) = 1$ and ζ_n primitive nth root of unity; coefficients in $\mathbb{C}$ or $\overline{\mathbb{F}}_q$ or $\mathbb{Z}[\zeta_n]$
- the GL_n-character variety:

$$\mathcal{M} := \{(A_i, B_i)_{i=1..g} \in \mathrm{GL}_n^{2g} \mid [A_1, B_1] \ldots [A_g, B_g] = \zeta_n^d I_n\} // \mathrm{PGL}_n$$

non-singular, affine

- the SL_n-character variety:

$$\check{\mathcal{M}} := \{(A_i, B_i)_{i=1..g} \in \mathrm{SL}_n^{2g} \mid [A_1, B_1] \ldots [A_g, B_g] = \zeta_n^d I_n\} // \mathrm{PGL}_n$$

non-singular, affine

Character variety for PGL_n

- $(\mathrm{GL}_1)^{2g}$ acts on $\mathcal{M}_\mathrm{B}$
- $\Gamma \cong (\mathbb{Z}_n)^{2g} \subset (\mathrm{GL}_1)^{2g}$ acts on $\check{\mathcal{M}}_\mathrm{B}$
- the PGL_n-character variety: $\hat{\mathcal{M}}_\mathrm{B} := \check{\mathcal{M}}_\mathrm{B}/\Gamma \cong \mathcal{M}_\mathrm{B}/(\mathrm{GL}_1)^{2g}$ is an affine orbifold

Character variety over a finite field

- Frobenius' character formula
  - for $\mathrm{GL}_n(\mathbb{F}_q) \rightsquigarrow \#\mathcal{M}_\mathrm{B}(\mathbb{F}_q) = \sum_{\chi \in \mathrm{Irr}(\mathrm{GL}_n(\mathbb{F}_q))} \frac{|\mathrm{GL}_n(\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-2}} \frac{\chi(\xi_n^d I_n)}{\chi(1)}$
  - for $\mathrm{SL}_n(\mathbb{F}_q) \rightsquigarrow \#\check{\mathcal{M}}_\mathrm{B}(\mathbb{F}_q) = \sum_{\chi \in \mathrm{Irr}(\mathrm{SL}_n(\mathbb{F}_q))} \frac{|\mathrm{SL}_n(\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-2}} \frac{\chi(\xi_n^d I_n)}{\chi(1)}$
  - for $\mathrm{PGL}_n(\mathbb{F}_q) \rightsquigarrow \#\hat{\mathcal{M}}_\mathrm{B}(\mathbb{F}_q) = \frac{\#\mathcal{M}_\mathrm{B}(\mathbb{F}_q)}{(q-1)^{2g}}$
- in all these cases the count is a polynomial in q

Serre-polynomial

- X complex algebraic variety
- (Deligne 1974) constructs weight filtration

  $W_0 \subset \cdots \subset W_k \subset \cdots \subset W_{2d} = H_c^d(X;\mathbb{Q})$

  with Hodge decomposition

$$W_k/W_{k-1}(H_c^d(X)) = \bigoplus_{i+j=k} H^{i,j}(W_k/W_{k-1}(H_c^d(X)))$$

- define *Serre-polynomial*

$$E(X;x,y) = \sum_{i,j,d} h^{i,j}(W_{i+j}/W_{i+j-1}(H_c^d(X)))(-1)^d x^i y^j$$

Polynomial-count varieties

Theorem (Katz 2008)

If X is a variety defined over $\mathbb{Z}$ and $\#\{X(\mathbb{F}_q)\} = E(q)$ is a polynomial in q, then

$$E(X(\mathbb{C}); x, y) = E(xy).$$

- as the character varieties $\mathcal{M}_{\mathrm{B}}$, $\check{\mathcal{M}}_{\mathrm{B}}$ and $\hat{\mathcal{M}}_{\mathrm{B}}$ all have polynomial-count Frobenius' formula gives their Serre-polynomial!
- e.g. for $n = 2$ $d = 1$ with $q = xy$
  $E(\hat{\mathcal{M}}_{\mathrm{B}}; x, y) = \#\{\hat{\mathcal{M}}_{\mathrm{B}}(\mathbb{F}_q)\} = \frac{\#\{\mathcal{M}_{\mathrm{B}}(\mathbb{F}_q)\}}{(q-1)^{2g}} =$

$$= (q^2-1)^{2g-2} + q^{2g-2}(q^2-1)^{2g-2} - \frac{1}{2}q^{2g-2}(q-1)^{2g-2} - \frac{1}{2}q^{2g-2}(q+1)^{2g-2}$$

$$E(\check{\mathcal{M}}_{\mathrm{B}}; x, y) = E(\hat{\mathcal{M}}_{\mathrm{B}}; x, y) + (2^{2g}-1)q^{2g-2}\left(\frac{(q-1)^{2g-2}-(q+1)^{2g-2}}{2}\right)$$

Betti version of Topological Mirror Test

- X non-singular algebraic variety$/\mathbb{C}$, Γ finite group acting on X
- define stringy Serre-polynomial of the orbifold X/Γ by

$$E^B_{st}(X/\Gamma; x, y) = \sum_{[\gamma]\in[\Gamma]} E(X^\gamma/C(\gamma), L^B_\gamma; x, y)(xy)^{F(\gamma)}$$

- motivating: (Kontsevich 1995) for $Y \to X/\Gamma$ crepant $\rightsquigarrow$ $E_{st}(X/\Gamma; x, y) = E(Y; x, y)$
- recall that the PGL_n-character variety $\hat{\mathcal{M}}_{\mathrm{B}} = \check{\mathcal{M}}_{\mathrm{B}}/\Gamma$ is an orbifold with $\Gamma \cong (\mathbb{Z}_n)^{2g}$

Conjecture (Hausel–Villegas 2004, Topological Mirror Test)

$$E(\check{\mathcal{M}}_{\mathrm{B}}; x, y) = E^B_{\mathrm{st}}(\hat{\mathcal{M}}_{\mathrm{B}}; x, y).$$

Example

- when $n = 2$, with $q = xy$

$$E(\check{\mathcal{M}}_{\mathrm{B}}; x, y) - E(\hat{\mathcal{M}}_{\mathrm{B}}; x, y) = (2^{2g} - 1)q^{2g-2}\left(\frac{(q-1)^{2g-2} - (q+1)^{2g-2}}{2}\right)$$

- $\check{\mathcal{M}}_{\mathrm{B}}^{\gamma}$ can be identified with $(\mathbb{C}^{\times})^{2g-2} \rightsquigarrow$

$$E(\check{\mathcal{M}}_{\mathrm{B}}^{\gamma}/\Gamma, L_{B,\gamma}; x, y) = \frac{(q-1)^{2g-2} - (q+1)^{2g-2}}{2}$$

- implies Betti version of Topological Mirror Test when $n = 2$
- similar argument settles $n = p$

Mirror symmetry for Langlands dual Hitchin systems

- by the non-Abelian Hodge theorem $\mathcal{M}_{\mathrm{B}} \overset{\textit{diff}}{\simeq} \mathcal{M}$ with moduli Higgs bundles $\rightsquigarrow$ Hitchin map $\chi : \mathcal{M}_{\mathrm{B}} \to \mathcal{A}$

- $$\begin{array}{ccc} \check{\mathcal{M}}_{\mathrm{B}} & & \hat{\mathcal{M}}_{\mathrm{B}} \\ & \check{\chi} \searrow \quad \swarrow \hat{\chi} & \\ & \mathcal{A} & \end{array}$$

  $\rightsquigarrow$ SYZ construction for mirror symmetric Calabi-Yau's
  $\rightsquigarrow$ $\check{\mathcal{M}}_{\mathrm{B}}$ and $\hat{\mathcal{M}}_{\mathrm{B}}$ could be considered mirror symmetric!

- Betti version of Topological Mirror Test is the agreement of Hodge numbers

- for $n = 2$ we proved Topological Mirror Test from certain patterns in $\mathit{Irr}(\mathrm{SL}_2(\mathbb{F}_q))$ vs. $\mathit{Irr}(\mathrm{GL}_2(\mathbb{F}_q))$
  due to (Schur 1907) and (Jordan 1907)

Jordan's character table of $\mathrm{PGL}_2(\mathbb{F}_q)$

The Binary Linear Fractional Group F_1 in the $GF[p^n]$, $p>2$, of all Determinants not Zero.

Below is given the table of group-characters.

N	1	1	1	1	$\frac{s-3}{2}$	$\frac{s-1}{2}$
χ_λ	1	1	s	s	$s+1$	$s-1$
χ_μ	1	1	0	0	1	-1
$\chi(R^{2a})$	1	1	1	1	$r^{2a}+r^{-2a}$	0
$\chi(S^{2b})$	1	1	-1	-1	0	$-t^{2b}-t^{-2b}$
$\chi(R^{2a+1})$	1	-1	1	-1	$r^{2a+1}+r^{-(2a+1)}$	0
$\chi(S^{2b+1})$	1	-1	-1	1	0	$-t^{2b+1}-t^{-(2b+1)}$

where r and t are the roots (except ± 1) of $r^{s-1}=1$ and $t^{s+1}=1$ respectively. As before $e=f$.

MICHIGAN COLLEGE OF MINES, HOUGHTON, MICH.

Schur's character table of $\mathrm{SL}_2(\mathbb{F}_q)$

2. Die Gruppe $\mathfrak{L}_{p^n}$, die durch die ganzen linearen Substitutionen

$$\xi_1 = \alpha\eta_1 + \beta\eta_2, \quad \xi_2 = \gamma\eta_1 + \delta\eta_2$$

gebildet wird, deren Determinante gleich 1 ist. — Die Ordnung der Gruppe

Die $s+4$ Charaktere von $\mathfrak{L}_s$ lassen sich in folgender Tabelle zusammenfassen:

	1	1	$\frac{s-3}{2}$	$\frac{s-1}{2}$	2	2
$\chi(E)$	1	s	$s+1$	$s-1$	$\frac{1}{2}(s+1)$	$\frac{1}{2}(s-1)$
$\chi(F)$	1	s	$(-1)^{\alpha}(s+1)$	$(-1)^{\beta}(s-1)$	$\frac{\varepsilon}{2}(s+1)$	$-\frac{\varepsilon}{2}(s-1)$
$\chi(P)$	1	0	1	-1	$\frac{1}{2}(1 \pm \sqrt{\varepsilon s})$	$\frac{1}{2}(-1 \pm \sqrt{\varepsilon s})$
$\chi(Q)$	1	0	1	-1	$\frac{1}{2}(1 \mp \sqrt{\varepsilon s})$	$\frac{1}{2}(-1 \mp \sqrt{\varepsilon s})$
$\chi(A^{a})$	1	1	$\varrho^{\alpha a} + \varrho^{-\alpha a}$	0	$(-1)^{a}$	0
$\chi(B^{b})$	1	-1	0	$-(\sigma^{\beta b} + \sigma^{-\beta b})$	0	$-(-1)^{b}$.

*) Vgl. *Dickson*, Linear Groups, Cap. XII.