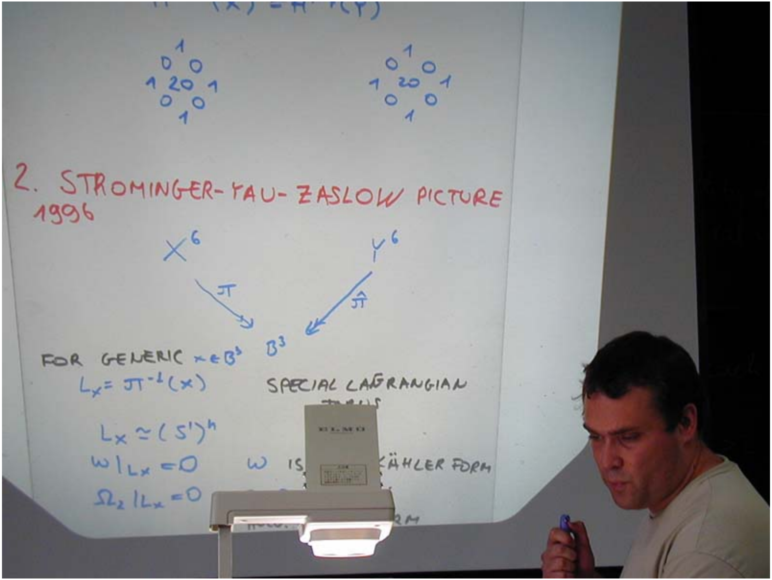
Mirror symmetry, Langlands duality and the Hitchin system I

Tamás Hausel

Royal Society URF at University of Oxford
http://www.maths.ox.ac.uk/∼hausel/talks.html

April 2010
Simons lecture series
Stony Brook

Talk with same title in RIMS, Kyoto 6 September 2001

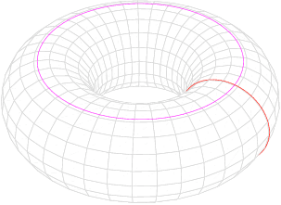


Mirror Symmetry

- phenomenon first arose in various forms in string theory
- mathematical predictions (Candelas-de la Ossa-Green-Parkes 1991)
- mathematically it relates the symplectic geometry of a Calabi-Yau manifold X^d to the complex geometry of its mirror Calabi-Yau Y^d
- first aspect is the *topological mirror test* $h^{p,q}(X) = h^{d-p,q}(Y)$
- compact hyperkähler manifolds satisfy $h^{p,q}(X) = h^{d-p,q}(X)$
- (Strominger-Yau-Zaslow 1996) suggests a geometrical construction how to obtain Y from X
- many predictions of mirror symmetry have been confirmed - no general understanding yet

Hodge numbers

- X^n non-singular complex algebraic variety
- $H^k(X;\mathbb{C}) := \frac{\{\alpha\in\Omega^k(X)|d\alpha=0\}}{\{\alpha\in\Omega^k(X)|\alpha=d\beta\}}$
- $H^{i,j}(X)$ closed forms α of type (i,j) i.e. locally $\alpha = \sum_{|I|=i,|J|=j} a_{I,J} dz_I \wedge d\bar{z}_J$ modulo exact forms
- by Hodge theory of harmonic forms $H^k(X;\mathbb{C}) = \oplus_{i+j=k} H^{i,j}(X)$
- Hodge numbers $h^{i,j} := \dim H^{i,j}(X)$
  Betti numbers $b_k := \dim H^k(X) = \sum_{i+j=k} h_{i,j}$
- symmetries:
  1. $h^{i,j} = h^{j,i}$ Hodge symmetry
  2. $h^{i,j} = h^{n-j,n-i}$ Poincaré duality
- example: Hodge diamond of a non-singular elliptic curve



$$h^{1,1} = 1$$

$$h^{1,0} = 1 \qquad\qquad h^{0,1} = 1$$

$$h^{0,0} = 1$$

Hodge diamonds of mirror Calabi-Yaus

Fermat quintic X

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & 101 & & 101 & & 1 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

$\hat{X} := X/(\mathbb{Z}_5)^3$

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 101 & & 0 & \\
1 & & 1 & & 1 & & 1 \\
 & 0 & & 101 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

K3 surface X

$$\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}$$

$\hat{X}$ mirror K3

$$\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}$$

Langlands duality

- the Langlands program aims to describe $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ via representation theory
- G reductive group, ${}^L\mathrm{G}$ its Langlands dual
- e.g ${}^L\mathrm{GL}_n = \mathrm{GL}_n$; ${}^L\mathrm{SL}_n = \mathrm{PGL}_n$, ${}^L\mathrm{PGL}_n = \mathrm{SL}_n$
- [Laumon 1987, Beilinson–Drinfeld 1995]
  Geometric Langlands conjecture on a complex curve X
  $\{\mathrm{G}\text{-local systems on } X\} \leftrightarrow \{\text{Hecke eigensheaves on } \mathrm{Bun}_{{}^L\mathrm{G}}(X)\}$
- [Kapustin–Witten 2006] deduces this from reduction of S-duality (electro-magnetic duality) in $N = 4$ SUSY YM in $4d$

Diffeomorphic spaces in non-Abelian Hodge theory

- C genus g curve; $\mathrm{G} = \mathrm{GL}_n(\mathbb{C})$ or $\mathrm{SL}_n(\mathbb{C})$

$$\mathcal{M}^d_{\mathrm{Dol}}(\mathrm{G}) := \left\{ \begin{array}{c} \text{moduli space of stable rank } n \\ \text{degree } d \text{ G-Higgs bundles } (E, \phi) \\ \text{i.e. } E \text{ rank } n \text{ degree d bundle on } C \\ \phi \in H^0(C, ad(E) \otimes K) \text{ Higgs field} \end{array} \right\}$$

$$\mathcal{M}^d_{\mathrm{DR}}(\mathrm{G}) := \left\{ \begin{array}{c} \text{moduli space of flat G-connections} \\ \text{on } C \setminus \{p\}, \text{ with holonomy } e^{\frac{2\pi i d}{n}} Id \text{ around } p \end{array} \right\}$$

- when $(d, n) = 1$ these are smooth non-compact varieties
- $\Gamma = Jac_C[n] \cong \mathbb{Z}_n^{2g}$ acts on $\mathcal{M}^d(\mathrm{SL}_n)$ by tensoring $\Rightarrow$ $\mathcal{M}^d(\mathrm{PGL}_n) := \mathcal{M}^d(\mathrm{SL}_n)/\Gamma$ is an orbifold

Theorem (Non-Abelian Hodge Theorem)

$$\mathcal{M}^d_{\mathrm{Dol}}(\mathrm{G}) \overset{diff}{\cong} \mathcal{M}^d_{\mathrm{DR}}(\mathrm{G})$$

Hitchin map

- the characteristic polynomial of $\phi \in H^0(C, End(E) \otimes K)$ $\chi(\phi) \in H^0(C, K) \oplus H^0(C, K^2) \oplus \cdots \oplus H^0(C, K^n)$ defines *Hitchin map*

$$\chi_{\mathrm{GL}_n} : \mathcal{M}^d_{\mathrm{Dol}}(\mathrm{GL}_n) \to \mathcal{A}_{\mathrm{GL}_n} = \oplus_{i=1}^n H^0(C, K^i)$$

$$\chi_{\mathrm{SL}_n} : \mathcal{M}^d_{\mathrm{Dol}}(\mathrm{SL}_n) \to \mathcal{A}_{\mathrm{SL}_n} = \oplus_{i=2}^n H^0(C, K^i)$$

$$\chi_{\mathrm{PGL}_n} : \mathcal{M}^d_{\mathrm{Dol}}(\mathrm{PGL}_n) \to \mathcal{A}_{\mathrm{PGL}_n} = \oplus_{i=2}^n H^0(C, K^i)$$

Theorem (Hitchin 1987, Nitsure 1991, Faltings 1993)

χ *is* proper *and a completely integrable Hamiltonian system.* $(\omega(X_{\chi_i}, X_{\chi_j}) = 0)$
Over a generic point $a \in \mathcal{A}$ *the fibre* $\chi^{-1}(a)$ *is a torsor for an Abelian variety.*

Theorem (Hausel,Thaddeus 2003)

For a generic $a \in \mathcal{A}_{\mathrm{SL}_n} \cong \mathcal{A}_{\mathrm{PGL}_n}$ *the fibres* $\chi^{-1}_{\mathrm{SL}_n}(a)$ *and* $\chi^{-1}_{\mathrm{PGL}_n}(a)$ *are torsors for* dual *Abelian varieties.*

$$\begin{array}{ccc} \mathcal{M}^d_{\mathrm{Dol}}(\mathrm{PGL}_n) & \leftarrow & \mathcal{M}^d_{\mathrm{Dol}}(\mathrm{SL}_n) \\ \downarrow \chi_{\mathrm{PGL}_n} & & \downarrow \chi_{\mathrm{SL}_n} \\ \mathcal{A}_{\mathrm{PGL}_n} & \cong & \mathcal{A}_{\mathrm{SL}_n}. \end{array}$$

$\Rightarrow \mathcal{M}^d_{\mathrm{DR}}(\mathrm{PGL}_n)$ *and* $\mathcal{M}^d_{\mathrm{DR}}(\mathrm{SL}_n)$ *satisfy the SYZ construction for a pair of mirror symmetric Calabi-Yau manifolds.*

Topological mirror tests

- (Deligne 1971) $\rightsquigarrow$ weight filtration for any complex algebraic variety X: $W_0 \subset \cdots \subset W_k \subset \cdots \subset W_{2d} = H_c^d(X;\mathbb{Q})$, plus a pure Hodge structure on W_k/W_{k-1} of weight k
- define $E(X;x,y) = \sum(-1)^d x^i y^j h^{i,j}\left(W_k/W_{k-1}(H_c^d(X,\mathbb{C}))\right)$
  $E_{st}^B(M/\Gamma) = \sum_{[\gamma]\in[\Gamma]} E(M^\gamma; L_\gamma^B)^{C(\gamma)}(uv)^{F(\gamma)}$
- if $Y \to X/\Gamma$ is crepant then (Kontsevich 1996) $\rightsquigarrow$
  $E_{st}(X/\Gamma;x,y) = E(Y;x,y)$

Conjecture (Hausel–Thaddeus 2003, "DR-TMS", "Dol-TMS")

For all $d, e \in \mathbb{Z}$, satisfying $(d,n) = (e,n) = 1$, we have

$$E_{\mathrm{st}}^{B^e}\left(\mathcal{M}_{\mathrm{DR}}^d(\mathrm{SL}_n(\mathbb{C}));x,y\right) = E_{\mathrm{st}}^{\hat{B}^d}\left(\mathcal{M}_{\mathrm{DR}}^e(\mathrm{PGL}_n(\mathbb{C}));x,y\right)$$

$$E_{\mathrm{st}}^{B^e}\left(\mathcal{M}_{\mathrm{Dol}}^d(\mathrm{SL}_n(\mathbb{C}));x,y\right) = E_{\mathrm{st}}^{\hat{B}^d}\left(\mathcal{M}_{\mathrm{Dol}}^e(\mathrm{PGL}_n(\mathbb{C}));x,y\right)$$

Example

- let $\check{\mathcal{M}}$ be moduli space of SL_2 parabolic Higgs bundles on elliptic curve E with one parabolic point
- $\mathbb{Z}_2$ acts on E and $\mathbb{C}$ as additive inverse $x \mapsto -x$
- $\check{\mathcal{M}} \to E \times \mathbb{C}/\mathbb{Z}_2$ blowing up; $\chi : \check{\mathcal{M}} \to \mathbb{C}/\mathbb{Z}_2 \cong \mathbb{C}$ is elliptic fibration with $\hat{D}_4$ singular fiber over 0
- $\Gamma = E[2] \cong \mathbb{Z}_2^2$ acts on $\check{\mathcal{M}}$ by multiplying on E
- $\hat{\mathcal{M}}$ the PGL_2 moduli space is $\check{\mathcal{M}}/\Gamma$ an orbifold, elliptic fibration over $\mathbb{C}$ with A_1 singular fiber with three $\mathbb{C}^2/\mathbb{Z}_2$-orbifold points on one of the components
- blowing up the three orbifold singularities is crepant gives $\check{\mathcal{M}}$
- the topological mirror test: $E_{st}(\hat{\mathcal{M}}; x, y) \overset{Kontsevich}{=} E(\check{\mathcal{M}}; x, y)$