

Mirror symmetry, Langlands duality and the Hitchin system

Tamás Hausel

Royal Society URF at University of Oxford
<http://www.maths.ox.ac.uk/~hausel/talks.html>

March 2010
Second International School on Geometry and Physics
Geometric Langlands and Gauge Theory
CRM Barcelona

$\pi(X) = \pi(Y)$

$$\begin{matrix} & 1 & & & \\ & 0 & 1 & & \\ 1 & 2 & 0 & & 1 \\ & 0 & 1 & & \\ & & & 1 & \end{matrix}$$

$$\begin{matrix} & & 1 & & \\ & 0 & 1 & 0 & \\ 1 & 2 & 0 & & 1 \\ & 0 & 1 & & \\ & & & 1 & \end{matrix}$$

2. STROMINGER-YAU-ZASLOW PICTURE

1996

X^6
 $\searrow \pi$

B^3

Y^6
 $\swarrow \tilde{\pi}$

FOR GENERIC $x \in B^3$

$L_x = \pi^{-1}(x)$

$L_x \simeq (S^1)^4$

$\omega|_{L_x} = 0$

$\Omega_2|_{L_x} = 0$

SPECIAL LAGRANGIAN

TORSUS

ω IS KÄHLER FORM

- phenomenon first arose in various forms in string theory
- mathematical predictions (Candelas-de la Ossa-Green-Parkes 1991)
- mathematically it relates the symplectic geometry of a Calabi-Yau manifold X^d to the complex geometry of its mirror Calabi-Yau Y^d
- first aspect is the *topological mirror test* $h^{p,q}(X) = h^{d-p,q}(Y)$
- compact hyperkähler manifolds satisfy $h^{p,q}(X) = h^{d-p,q}(X)$
- (Kontsevich 1994) suggests *homological mirror symmetry* $\mathcal{D}^b(\text{Fuk}(X, \omega)) \cong \mathcal{D}^b(\text{Coh}(Y, I))$
- (Strominger-Yau-Zaslow 1996) suggests a geometrical construction how to obtain Y from X
- many predictions of mirror symmetry have been confirmed - no general understanding yet

Hodge diamonds of mirror Calabi-Yaus

Fermat quintic X

			1			
		0		0		
	0		1		0	
1		101		101		1
	0		1		0	
		0		0		
			1			

$\hat{X} := X/(\mathbb{Z}_5)^3$

				1		
		0		0		
	0		101		0	
1		1		1		1
	0		101		0	
		0		0		
			1			

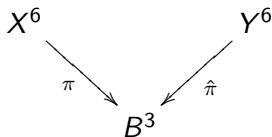
K3 surface X

			1			
		0		0		
1		20		20		1
		0		0		
			1			

\hat{X} mirror K3

				1		
		0		0		
1		20		20		1
		0		0		
			1			

- X CY 3-fold
- Y mirror CY 3-fold
- B is 3-dimensional real manifold - mostly S^3



- π and $\hat{\pi}$ are special Lagrangian fibrations
- for generic $x \in B^3$
 $L_x = \pi^{-1}(x) \cong T^3$ and $\hat{L}_x = \hat{\pi}^{-1}(x) \cong T^3$ are dual special Lagrangian tori
- generically Y^6 can be thought of as the moduli space of flat $U(1)$ connections on a generic fiber L_x (a.k.a. *D-branes*)

- the Langlands program aims to describe $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ via representation theory
- G reductive group, ${}^L G$ its Langlands dual
- e.g. ${}^L \text{GL}_n = \text{GL}_n$; ${}^L \text{SL}_n = \text{PGL}_n$, ${}^L \text{PGL}_n = \text{SL}_n$
- [Langlands 1967] conjectures that $\{\text{homs } \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow G(\mathbb{C})\} \leftrightarrow \{\text{automorphic reps of } {}^L G(\mathcal{A}_{\mathbb{Q}})\}$
- $G = \text{GL}_1 \rightsquigarrow$ class field theory
 $G = \text{GL}_2 \rightsquigarrow$ Shimura-Taniyama-Weil
- function field version: replace \mathbb{Q} with $\mathbb{F}_q(X)$, where X/\mathbb{F}_q is algebraic curve
- [Ngô, 2008] proves fundamental lemma for $\mathbb{F}_q(X) \rightsquigarrow \text{FL}$ for \mathbb{Q}
- geometric version: replace $\mathbb{F}_q(X)$ with $\mathbb{C}(X)$ for X/\mathbb{C}
- [Laumon 1987, Beilinson–Drinfeld 1995]
Geometric Langlands conjecture
 $\{G\text{-local systems on } X\} \leftrightarrow \{\text{Hecke eigensheaves on } \text{Bun}_{{}^L G}(X)\}$
- [Kapustin–Witten 2006] deduces this from reduction of S-duality (electro-magnetic duality) in $N = 4$ SUSY YM in $4d$

- Hamiltonian system: (X^{2d}, ω) symplectic manifold
 $H : X \rightarrow \mathbb{R}$ Hamiltonian function X_H Hamiltonian vector field
($dH = \omega(X_H, \cdot)$)
- $f : X \rightarrow \mathbb{R}$ is a *first integral* if $X_H f = \omega(X_f, X_H) = 0$
- the Hamiltonian system is *completely integrable* if there is
 $f = (H = f_1, \dots, f_d) : X \rightarrow \mathbb{R}^d$ generic such that
 $\omega(X_{f_i}, X_{f_j}) = 0$
- the generic fibre of f has an action of $\mathbb{R}^d = \langle X_{f_1}, \dots, X_{f_d} \rangle \rightsquigarrow$
when f is proper generic fibre is a torus $(S^1)^d$
- examples include: Euler and Kovalevskaya tops and the spherical pendulum
- algebraic version when replacing \mathbb{R} by $\mathbb{C} \rightsquigarrow$ many examples can be formulated as a version of the *Hitchin system*
- a Hitchin system is associated to a complex curve C and a complex reductive group G
- it arose in the study [Hitchin 1987] of the 2-dimensional reduction of the Yang-Mills equations

- In these lectures we will discuss the mirror symmetry proposal of [Hausel–Thaddeus 2003]:
"Hitchin systems for Langlands dual groups satisfy Strominger-Yau-Zaslow, so could be considered mirror symmetric; in particular they should satisfy the *topological mirror tests*:"

- $$\begin{array}{ccc} \mathcal{M}_{\text{DR}}^d(\text{SL}_n) & & \mathcal{M}_{\text{DR}}^e(\text{PGL}_n) \\ & \searrow \check{\chi} & \swarrow \hat{\chi} \\ & \mathcal{A}^0 & \end{array}$$

Conjecture (Hausel–Thaddeus 2003, "Topological mirror test")

For all $d, e \in \mathbb{Z}$, satisfying $(d, n) = (e, n) = 1$, we have

$$E_{\text{st}}^{B^e} \left(\mathcal{M}_{\text{DR}}^d(\text{SL}_n); x, y \right) = E_{\text{st}}^{\hat{B}^d} \left(\mathcal{M}_{\text{DR}}^e(\text{PGL}_n); x, y \right).$$

- C smooth complex projective curve of genus $g > 1$
- fix integers $n > 0$ and $d \in \mathbb{Z}$ always assume $(d, n) = 1$.
- $\mathcal{N}^d :=$ moduli space of isomorphism classes of semi-stable rank n degree d vector bundles on C
- constructed using geometric invariant theory (GIT) or gauge theory
- vector bundle E is called *semi-stable* (*stable*) if every proper subbundle F satisfies

$$\mu(F) = \frac{\deg(F)}{\mathrm{rk}(F)} \stackrel{(<)}{\leq} \mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)}$$

- when $(d, n) = 1$ semi-stability \Leftrightarrow stability \rightsquigarrow
 \mathcal{N}^d is a non-singular projective fine moduli space

- $\det : \mathcal{N}^d \rightarrow \text{Jac}^d(C)$
 $[E] \mapsto \Lambda^n(E)$
- fix $\Lambda \in \text{Jac}^d(C)$ and let $\check{\mathcal{N}}^\Lambda := \det^{-1}(\Lambda) \subset \mathcal{N}^d$
the moduli space of (twisted) SL_n bundles on C
- $\check{\mathcal{N}}^\Lambda$ does not depend on the choice of $\Lambda \in \text{Jac}^d(C)$ just write
 $\check{\mathcal{N}}^d := \check{\mathcal{N}}^\Lambda$
- when $(d, n) = 1 \rightsquigarrow \check{\mathcal{N}}^d$ is non-singular and projective
- $\text{Pic}^0(C) = \text{Jac}^0(C)$ acts on \mathcal{N}^d via $(L, E) \mapsto L \otimes E$. define

$$\hat{\mathcal{N}}^d := \mathcal{N}^d / \text{Pic}^0(C)$$

the moduli space of degree d PGL_n bundles on C

- $\Gamma := \text{Pic}^0(C)[n] \cong \mathbb{Z}_n^{2g} \subset \text{Pic}^0(C)$ acts on $\hat{\mathcal{N}}^d$ and clearly
 $\hat{\mathcal{N}}^d = \check{\mathcal{N}}^d / \Gamma \rightsquigarrow \hat{\mathcal{N}}^d$ is a projective orbifold.

- The cohomologies $H^*(\mathcal{N}^d)$, $H^*(\check{\mathcal{N}}^d)$ and $H^*(\hat{\mathcal{N}}^d)$ are well understood.
- [Harder–Narasimhan 1975] obtained recursive formulae for $\#\mathcal{N}(\mathbb{F}_q) \rightsquigarrow$ formula for Betti numbers via the Weil conjectures [Deligne 1974]
- [Atiyah–Bott 1981] gave different gauge-theoretic proof

Theorem (Harder–Narasimhan, 1975)

The finite group Γ acts trivially on $H^(\check{\mathcal{N}}^d)$.
In particular $H^*(\check{\mathcal{N}}^d) \cong H^*(\hat{\mathcal{N}}^d)$.*

- proof by showing $\#\check{\mathcal{N}}^d(\mathbb{F}_q) = \#\hat{\mathcal{N}}^d(\mathbb{F}_q)$
- [Hitchin, 1987] \Rightarrow false for moduli space of SL_2 Higgs bundles \rightsquigarrow non-triviality of our topological mirror tests

The Hitchin map - GL_n

- $T^*\mathcal{N}$ is a (non-projective) algebraic symplectic variety
- the ring $\mathbb{C}[T^*\mathcal{N}]$ is known to be finitely-generated
- the affinization of $T^*\mathcal{N}$ gives the GL_n Hitchin map.

$$\chi : T^*\mathcal{N} \rightarrow \mathcal{A} := \text{Spec}(\mathbb{C}[T^*\mathcal{N}])$$

- deformation theory $\rightsquigarrow T_{[E]}\mathcal{N} = H^1(C, \text{End}(E))$
Serre duality $\Rightarrow T_{[E]}^*\mathcal{N} = H^0(C, \text{End}(E) \otimes K)$
- $\phi \in H^0(C, \text{End}(E) \otimes K)$ is a *Higgs field*
locally "a matrix of one-forms on the curve"
- let $(E, \phi) \in T^*\mathcal{N}$ its characteristic polynomial
 $\chi(\phi) = t^n + a_1 t^{n-1} + \dots + a_n$ where $a_i \in H^0(K^n)$
- $$\begin{aligned} \chi : T^*\mathcal{N} &\rightarrow \mathcal{A} := \bigoplus_{i=1}^n H^0(K^i) \\ (E, \phi) &\mapsto (a_1, a_2, \dots, a_n) \end{aligned}$$
- The affine space \mathcal{A} is called the *Hitchin base*.

- for SL_n

$$T_{[E]}^* \check{\mathcal{N}}^d = H^0(\text{End}_0(E) \otimes K)$$

that is, a covector at E is given by a *trace free* Higgs field.

- the SL_n Hitchin base is

$$\check{\mathcal{A}} = \mathcal{A}^0 := \bigoplus_{i=2}^n H^0(C, K^i).$$

- the SL_n Hitchin map

$$\check{\chi} : T^* \check{\mathcal{N}}^d \rightarrow \mathcal{A}^0.$$

- the action of $\Gamma = \text{Pic}^0(C)[n]$ on $T^* \check{\mathcal{N}}$ is along the fibers of $\check{\chi}$
 $\Rightarrow \check{\chi}$ descends to the quotient
- the PGL_n Hitchin map:

$$\hat{\chi} : (T^* \check{\mathcal{N}}) / \Gamma \rightarrow \hat{\mathcal{A}} = \mathcal{A}^0.$$

The Hitchin map is an integrable system

- recall that $T^*\mathcal{N}$ is an algebraic symplectic variety
- with canonical Liouville symplectic structure
- as the Hitchin map only depends on the cotangent direction
 \leadsto

Theorem (Hitchin, 1987)

- $\omega(X_{\chi_i}, X_{\chi_j}) = 0$ for any two $\chi_i, \chi_j \in \mathbb{C}[T^*\mathcal{N}]$ coordinate functions.
 - $\dim(\mathcal{A}) = \dim(\mathcal{N}) = \dim(T^*\mathcal{N})/2$
 - generic fibres of χ are open subsets of abelian varieties
- $\leadsto \chi$ is an algebraically completely integrable Hamiltonian system.
- Need to projectivize χ to complete the generic fibres to abelian varieties (compact tori)

Proper Hitchin map

- $(E, \phi) \in T^*\mathcal{N} \rightsquigarrow E$ is stable; to projectivize χ we need to allow E to become unstable.
- A *Higgs bundle* is a pair (E, ϕ) where E is a vector bundle on C and $\phi \in H^0(C, \text{End}(E) \otimes K)$ is a *Higgs field*.
- a Higgs bundle (E, ϕ) is *(semi-)stable* if for every ϕ -invariant proper subbundle F we have $\mu(F) \stackrel{(\leq)}{<} \mu(E)$
- \mathcal{M}^d the moduli space of (semi-)stable Higgs bundles, a non-singular quasi-projective and symplectic variety, containing $T^*\mathcal{N} \subset \mathcal{M}^d$ as an open dense subvariety
- extend $\chi : \mathcal{M}^d \rightarrow \mathcal{A}$ in the obvious way

Theorem (Hitchin 1987, Nitsure 1991, Faltings 1993)

χ is a proper algebraically completely integrable Hamiltonian system. Its generic fibres are abelian varieties.

- as $\text{codim}(\mathcal{M}^d \setminus T^*\mathcal{N}^d) \geq 2 \stackrel{\text{Hartogs}}{\Rightarrow} \mathbb{C}[\mathcal{M}^d] \cong \mathbb{C}[T^*\mathcal{N}^d] \Rightarrow$
by the Theorem
 $\mathcal{A} \cong \text{Spec}(\mathbb{C}[\mathcal{M}^d]) \cong \text{Spec}(\mathbb{C}[T^*\mathcal{N}^d])$

- fix $\Lambda \in \text{Jac}^d(C)$
- E vector bundle on C with determinant Λ
- $\phi \in H^0(\text{End}_0(E) \otimes K)$ is trace-free Higgs field
- then (E, ϕ) is an SL_n -Higgs bundle
- $\check{\mathcal{M}}^\Lambda \subset \mathcal{M}^d$ moduli space of (semi-)stable SL_n -Higgs bundles
- $\check{\mathcal{M}}^\Lambda$ is independent of Λ denote $\check{\mathcal{M}}^d := \check{\mathcal{M}}^\Lambda$
- $\check{\mathcal{M}}^d$ is a non-singular quasi-projective and symplectic variety
- characteristic polynomial of ϕ gives SL_n -Hitchin system

$$\check{\chi} : \check{\mathcal{M}}^d \rightarrow \mathcal{A}^0 := \bigoplus_{i=2}^n H^0(C; K^i)$$

- $\check{\chi}$ is proper and a completely integrable system

- $T^* \mathrm{Pic}^0(C) = \mathrm{Pic}^0(C) \times H^0(C, K)$ is a group; it acts on \mathcal{M}^d by $(L, \varphi)(E, \phi) \mapsto (L \otimes E, \varphi + \phi)$
- \rightsquigarrow action of $\Gamma = \mathrm{Pic}^0[n]$ on $\check{\mathcal{M}}^d$
- $\hat{\mathcal{M}}^d = \mathcal{M}^d / T^* \mathrm{Pic}^0(C) \cong \chi^{-1}(\mathcal{A}^0) / \mathrm{Pic}^0(C) \cong \check{\mathcal{M}} / \Gamma$
- $\hat{\mathcal{M}}^d$, the PGL_n *Higgs moduli space*, is an orbifold
- the Γ action is along the fibers of $\check{\chi} \rightsquigarrow \mathrm{PGL}_n$ *Hitchin map*

$$\hat{\chi}: \hat{\mathcal{M}}^d = \check{\mathcal{M}}^d / \Gamma \rightarrow \mathcal{A}^0$$

-

$$\begin{array}{ccc} \check{\mathcal{M}}^d & & \hat{\mathcal{M}}^e \\ & \searrow \check{\chi} & \swarrow \hat{\chi} \\ & \mathcal{A}^0 & \end{array}$$

- will show generic fibers are dual Abelian varieties; which are complex Lagrangian due to integrable system
- changing complex structure will lead to special Lagrangian fibrations; and so to SYZ

- let (E, ϕ) be a Higgs bundle such that $\chi(\phi) = a \in \mathcal{A}$ has the form

$$a = t^n + a_1 t^{n-1} + \cdots + a_n,$$

where $a_i \in H^0(K^i)$.

- What should be the spectrum of the Higgs field ϕ ?
- at $p \in C$ the Higgs field $\phi_p : E_p \rightarrow E_p \otimes K_p$
- eigenvalue ν_p of ϕ_p satisfies $\exists v \in E_p - 0 : \phi_p(v) = \nu_p v$. \leadsto must have $\nu_p \in K_p$
- let X denote the total space of K then $C_a := \cup_{p \in C} \nu_p^i \subset X$, the set of all eigenvalues of the Higgs field \leadsto *spectral curve*
- scheme structure on C_a ?
- tautological section $\lambda \in H^0(X, \pi^* K)$ satisfying $\lambda(x) = x$
- $s_a := \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n \in H^0(X, \pi^* K^n)$
- $C_a := s_a^{-1}(0) \subset X$ spectral curve
 $\pi_a : C_a \rightarrow C$ spectral cover of degree n

Generic fibres of the Hitchin map

- assume C_a is smooth $\Leftrightarrow a \in \mathcal{A}_{reg}$; $(E, \phi) \in \chi^{-1}(a) =: \mathcal{M}_a$
- if $\nu_p \in C_a \subset X$ then $L_{\nu_p} \subset \pi_a^*(E)$ ν_p -eigenspace in $E_p \rightsquigarrow$
 $L := \ker(\lambda Id_E - \pi_a^*(\phi)) \subset \pi_a^*(E)$ subsheaf rank 1 \rightsquigarrow invertible
as C_a is smooth such that
 $\pi_*(L(\Delta)) = E$ (eigenspace decomposition of ϕ)
- starting with a line bundle $M \in \text{Jac}^d(C_a)$ we construct
 $E = \pi_*(M)$ rank n degree $d' = d - n(n-1)(g-1)$ torsion
free \rightsquigarrow locally free and Higgs field
 $\phi := \pi_*(\lambda) : \pi_*(M) \rightarrow \pi_*(M) \otimes K$ pushing forward the
tautological map $\lambda : M \rightarrow M \otimes \pi^*(K)$
- by definition λ solves the characteristic polynomial a on $C_a \rightsquigarrow$
so will $\phi \rightsquigarrow$ by Cayley-Hamilton $\chi(\phi) = a$
- the spectral curve of a proper Higgs subbundle of
 $(E, \phi) = (\pi_*(M), \pi_*(\lambda))$ would be a 1-dimensional proper
subscheme of $C_a \Rightarrow (E, \phi)$ is stable

Theorem (Hitchin 1987, Beauville-Narasimhan-Ramanan 1989)

For $a \in \mathcal{A}_{reg}$ we have $\mathcal{M}_a^{d'} \cong \text{Jac}^d$.

Generic fibers for SL_n and PGL_n -Hitchin map

- recall (E, ϕ) SL_n -Higgs bundle if $\text{tr}(\phi) = 0$ and $\det(E) = \Lambda \in \text{Jac}^{d'}(C)$
- define $\text{Prym}^d(C) \subset \text{Jac}^d(C_a)$ by

$$L \in \text{Prym}^d(C_a) \Leftrightarrow \det \pi_*(L) = \Lambda$$

- if $a \in \mathcal{A}_{reg}^0$ the SL_n -Hitchin fibre satisfies

$$\check{\mathcal{M}}_a := \check{\chi}^{-1}(a) \cong \text{Prym}^d(C_a).$$

- for PGL_n we have $\hat{\mathcal{M}}_a := \hat{\chi}^{-1}(a) \cong \check{\mathcal{M}}_a/\Gamma \cong \text{Prym}^d(C_a)/\Gamma$ makes sense since for $L_\gamma \in \text{Pic}(C)[n]$ we have $\det(\pi_*(\pi^*(L_\gamma) \otimes L)) = \det(L_\gamma \otimes \pi_*(L)) = L_\gamma^n \otimes \det(\pi_*L) = \det(\pi_*L)$.
- alternatively $\hat{\mathcal{M}}_a = \mathcal{M}_a/\text{Pic}^0(C) \cong \text{Jac}^d(C_a)/\text{Pic}^0(C)$
- where $\text{Pic}^0(C)$ acts on $\text{Jac}^d(C_a)$ via the homomorphism $\pi_a^* : \text{Pic}^0(C) \rightarrow \text{Pic}^0(C_a)$

Symmetries of the GL_n and PGL_n Hitchin fibration

- for GL_n : fix $a \in \mathcal{A}_{reg}$
- tensor product gives a simply transitive action of $\text{Pic}^0(C_a)$ on $\text{Jac}^d(C_a)$
- $\leadsto \mathcal{M}_a$ is a torsor for $P_a := \text{Pic}^0(C_a)$
- for PGL_n : fix $a \in \mathcal{A}_{reg}^0$

$$\hat{\mathcal{M}}_a = \mathcal{M}_a / \text{Pic}^0(C)$$

is a torsor for the quotient $\hat{P}_a := P_a / \text{Pic}^0(C)$ abelian variety

Symmetries of the SL_n Hitchin fibration

- recall the spectral cover map $\pi : C_a \rightarrow C$
- for an abelian variety A the dual $\hat{A} := \text{Pic}^0(A)$

Definition

For $a \in \mathcal{A}_{reg}^0$ the *norm map* $Nm_{C_a/C} : \text{Pic}^0(C_a) \rightarrow \text{Pic}^0(C)$ is defined in any of the following three equivalent ways:

- 1 D divisor on C_a , $Nm_{C_a/C}(\mathcal{O}(D)) = \mathcal{O}(\pi_* D)$
- 2 For $L \in \text{Pic}^0(C_a)$ define
$$Nm_{C_a/C}(L) = \det(\pi_*(L)) \otimes \det^{-1}(\pi_* \mathcal{O}_{C_a}).$$
- 3 the norm map is the dual of the pull-back map
 $\pi_a^* : \text{Pic}^0(C) \rightarrow \text{Pic}^0(C_a)$, that is
$$Nm_{C_a/C} = \check{\pi} : \text{Pic}^0(C_a) \cong \check{\text{Pic}}^0(C_a) \rightarrow \check{\text{Pic}}^0(C) \simeq \text{Pic}^0(C).$$

- the Prym variety $\text{Prym}^0(C_a) := \ker(Nm_{C_a/C})$ acts on $\text{Prym}^d(C_a) = \check{\mathcal{M}}_a \rightsquigarrow \check{\mathcal{M}}_a$ is a torsor for $\check{P}_a := \text{Prym}^0(C_a)$.
- for PGL_n : $\hat{\mathcal{M}}_a$ is a torsor for $\hat{P}_a = \text{Pic}^0(C_a)/\text{Pic}^0(C) \cong \text{Prym}^0(C_a)/\Gamma \cong \check{P}_a/\Gamma$

Duality of the Hitchin fibres

- short exact sequence of abelian varieties:

$$0 \rightarrow \mathrm{Prym}^0(C_a) \hookrightarrow \mathrm{Pic}^0(C_a) \xrightarrow{\mathrm{Nm}_{C_a/C}} \mathrm{Pic}(C) \rightarrow 0$$

- the dual sequence is

$$0 \leftarrow \check{\mathrm{Prym}}^0(C_a) \leftarrow \mathrm{Pic}^0(C_a) \xleftarrow{\pi^*} \mathrm{Pic}(C) \leftarrow 0,$$

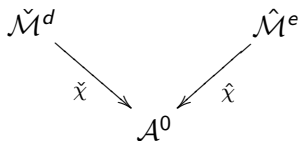
- $\check{\sim} \check{P}_a = \mathrm{Pic}^0(C_a)/\mathrm{Pic}(C) = \hat{P}_a, \Rightarrow \check{P}_a$ and \hat{P}_a are dual abelian varieties

Theorem (Hausel-Thaddeus, 2003)

For a regular $a \in \mathcal{A}_{\mathrm{reg}}^0$ $\check{\mathcal{M}}_a$ and $\hat{\mathcal{M}}_a$ are torsors for dual Abelian varieties (namely \check{P}_a and \hat{P}_a).

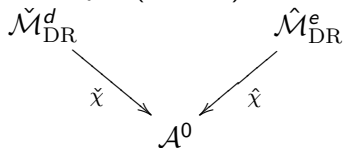
Strominger-Yau-Zaslow for $\check{\mathcal{M}}_{\text{DR}}$ and $\hat{\mathcal{M}}_{\text{DR}}$

-



- generic fibers are torsors for dual Abelian varieties
- as $\check{\chi}$ and $\hat{\chi}$ are integrable systems \Rightarrow the fibers are complex Lagrangian (i.e. $\omega^c = \omega_J + i\omega_K$ is zero on the fibers)
- [Hitchin, 1987] shows that $\check{\mathcal{M}}$ is hyperkähler and $(\check{\mathcal{M}}, J)$ is the moduli space $\check{\mathcal{M}}_{\text{DR}}$ of (twisted) flat SL_n -connections on C

- \rightsquigarrow



- the fibers of $\check{\chi}$ on $\check{\mathcal{M}}_{\text{DR}}$ now are special Lagrangian because both ω_J and $\text{Im}((\omega_K + i\omega_I)^{2d})$ restrict to zero on the fibers
- Strominger-Yau-Zaslow is satisfied for $\check{\mathcal{M}}_{\text{DR}}$ and $\hat{\mathcal{M}}_{\text{DR}}$!

- (Deligne 1972) constructs weight filtration $W_0 \subset \cdots \subset W_k \subset \cdots \subset W_{2d} = H_c^d(X; \mathbb{Q})$ for any complex algebraic variety X , plus a pure Hodge structure on W_k/W_{k-1} of weight k
- we say that the weight filtration is *pure* when $W_k/W_{k-1}(H_c^i(X)) \neq 0 \Rightarrow k = i$; examples include smooth projective varieties, $\hat{\mathcal{M}}^d$ and $\hat{\mathcal{M}}_{\text{DR}}^d$
- define $E(X; x, y) := \sum_{i,j,d} (-1)^d x^i y^j h^{i,j} (W_k/W_{k-1}(H_c^d(X, \mathbb{C})))$
- basic properties:
 - additive - if $X_i \subset X$ locally closed s.t. $\dot{\cup} X_i = X$ then $E(X; x, y) = \sum E(X_i; x, y)$
 - multiplicative - $F \rightarrow E \rightarrow B$ locally trivial in the Zariski topology $E(E; x, y) = E(B; x, y)E(F; x, y)$
- when weight filtration is pure then $E(X; -x, -y) = \sum_{p,q} h^{p,q}(H_c^{p+q}(X)) x^p y^q$ is the Hodge $E(X; t, t)$ is the Poincaré polynomial

- let finite group Γ act on a non-singular complex variety M
- $E_{st}(M/\Gamma; x, y) := \sum_{[\gamma] \in [\Gamma]} E(M_\gamma/C(\gamma); x, y)(xy)^{F(\gamma)}$
stringy E-polynomial
- $F(\gamma)$ is the fermionic shift, defined as $F(\gamma) = \sum w_i$, where γ acts on $TX|_{X_\gamma}$ with eigenvalues $e^{2\pi i w_i}$, $w_i \in [0, 1)$
- $F(\gamma)$ is an integer when M is CY and Γ acts trivially on K_M
- motivating property [Kontsevich 1995] if $f : X \rightarrow M/\Gamma$ crepant resolution $\Leftrightarrow K_X = f^* K_{M/\Gamma}$ then
 $E(X; x, y) = E_{st}(M/\Gamma; x, y)$
- if B is a Γ -equivariant flat $U(1)$ -gerbe on M , then on each \mathcal{M}_γ we get an automorphism of $B|_{\mathcal{M}_\gamma} \rightsquigarrow C(\gamma)$ -equivariant local system $L_{B,\gamma}$
- we can define
 $E_{st}^B(M/\Gamma; x, y) := \sum_{[\gamma] \in [\Gamma]} E(M_\gamma, L_{B,\gamma}; x, y)^{C(\gamma)}(xy)^{F(\gamma)}$
stringy E-polynomial twisted by a gerbe

Topological mirror symmetry conjecture - unravelled

Conjecture (Hausel–Thaddeus, 2003)

$(d, n) = (e, n) = 1$; \hat{B} the canonical Γ -equivariant gerbe on $\check{\mathcal{M}}_{\text{DR}}^e$

$$E(\check{\mathcal{M}}_{\text{DR}}^d) = E_{st}^{\hat{B}^e}(\hat{\mathcal{M}}_{\text{DR}}^e) \Leftrightarrow E(\check{\mathcal{M}}^d) = E_{st}^{\hat{B}^e}(\hat{\mathcal{M}}^e)$$

- Theorem for $n = 2, 3$ using [Hitchin 1987] and [Gothen 1994].
- as Γ acts on $H^*(\check{\mathcal{M}}^d)$ we have \rightsquigarrow
 $H^*(\check{\mathcal{M}}^d) \cong \bigoplus_{\kappa \in \hat{\Gamma}} H_{\kappa}^*(\check{\mathcal{M}}^d) \rightsquigarrow$

$$E(\check{\mathcal{M}}^d) = \sum_{\kappa \in \hat{\Gamma}} E_{\kappa}(\check{\mathcal{M}}^d) = E_0(\check{\mathcal{M}}^d) + \overbrace{\sum_{\kappa \in \hat{\Gamma}^*} E_{\kappa}(\check{\mathcal{M}}^d)}^{\text{variant}}$$

$$E_{st}^{B^d}(\hat{\mathcal{M}}^e) = \sum_{\gamma \in \Gamma} E(\check{\mathcal{M}}_{\gamma}^e, L_{B, \gamma})^{\Gamma} = E(\check{\mathcal{M}}^d)^{\Gamma} + \underbrace{\sum_{\gamma \in \Gamma^*} E(\check{\mathcal{M}}_{\gamma}^d / \Gamma, L_{B^d, \gamma})}_{\text{stringy}}$$

- $\Gamma \cong H^1(C, \mathbb{Z}_n)$ and wedge product induces $w : \Gamma \cong \hat{\Gamma}$
- refined Topological Mirror Test for $w(\gamma) = \kappa$:

$$E_{\kappa}(\check{\mathcal{M}}^d) = E(\check{\mathcal{M}}_{\gamma}^d / \Gamma, L_{B, \gamma})(xy)^{F(\gamma)}$$

Example SL_2

- fix $n = 2$ $d = 1$
- $\mathbb{T} := \mathbb{C}^\times$ acts on $\check{\mathcal{M}}$ by $\lambda \cdot (E, \phi) \mapsto (E, \lambda \cdot \phi) \xrightarrow{\text{Morse}} \check{\mathcal{M}}$

$$H^*(\check{\mathcal{M}}) = \bigoplus_{F_i \subset \check{\mathcal{M}}/\mathbb{T}} H^{*+\mu_i}(F_i) \quad \text{as } \Gamma\text{-modules}$$

- $F_0 = \check{\mathcal{N}}$ where $\phi = 0$; then [Harder–Narasimhan 1975] \Rightarrow
 $H^*(F_0)$ is trivial Γ -module
- for $i = 1, \dots, g-1$

$$F_i = \{(E, \phi) \mid E \cong L_1 \oplus L_2, \phi = \begin{pmatrix} 0 & 0 \\ \varphi & 0 \end{pmatrix}, \varphi \in H^0(L_1^{-1}L_2K)\}$$

$\leadsto F_i \rightarrow S^{2g-2i-1}(\mathbb{C})$ Galois cover with Galois group Γ

Theorem (Hitchin 1987)

The Γ action on $H^(F_i)$ is only non-trivial in the middle degree $2g - 2i - 1$. For $\kappa \in \hat{\Gamma}^*$ we have*

$$\dim H_{\kappa}^{2g-2i-1}(F_i) = \binom{2g-2}{2g-2i-1}.$$

Example PGL₂

- $\gamma \in \Gamma = \text{Pic}^0(C)[2] \rightsquigarrow C_\gamma \xrightarrow{2:1} C$ with Galois group \mathbb{Z}_2

-

$$\begin{array}{ccc}
 \mathcal{M}(\text{GL}_1, C_\gamma) \cong T^* \text{Jac}^d(C_\gamma) & \xrightarrow{\text{push-forward}} & \mathcal{M}^d \supset \check{\mathcal{M}}^d \\
 \parallel & & \downarrow \det \\
 T^* \text{Jac}^d(C_\gamma) & \xrightarrow{N_m(C_\gamma/C)} & T^* \text{Jac}^d(C) \ni (\Lambda, 0)
 \end{array}$$

- let $\check{\mathcal{M}}(\text{GL}_1, C_\gamma) := N_m(C_\gamma/C)^{-1}(\Lambda, 0)$ endoscopic H_γ -Higgs moduli space
- after [Narasimhan–Ramanan, 1975]
 $\check{\mathcal{M}}_\gamma = \check{\mathcal{M}}(\text{GL}_1, C_\gamma)/\mathbb{Z}_2 \cong T^* \text{Prym}^d(C_\gamma/C)$
- can calculate $\dim H^{2g-2i+1}(\check{\mathcal{M}}_\gamma/\Gamma, L_{\hat{B},\gamma}) = \binom{2g-2}{2g-2i-1}$
 and 0 otherwise

Theorem (Hausel–Thaddeus, 2003)

when $n = 2$ and $\kappa = w(\gamma)$

$$E_\kappa(\check{\mathcal{M}}; u, v) = E(\check{\mathcal{M}}_\gamma/\Gamma; L_{B,\gamma}, u, v)(uv)^{F(\gamma)}$$

Character varieties

- the GL_n -character variety:

$$\mathcal{M}_B^d := \{(A_i, B_i)_{i=1..g} \in GL_n^{2g} \mid [A_1, B_1] \dots [A_g, B_g] = \zeta_n^d I_n\} // PGL_n$$

non-singular, affine

- the SL_n -character variety:

$$\check{\mathcal{M}}_B^d := \{(A_i, B_i)_{i=1..g} \in SL_n^{2g} \mid [A_1, B_1] \dots [A_g, B_g] = \zeta_n^d I_n\} // PGL_n$$

non-singular, affine

- for PGL_n note that $(\mathbb{C}^\times)^{2g}$ acts on \mathcal{M}_B^d and

$$\Gamma \cong (\mathbb{Z}_n)^{2g} \subset (\mathbb{C}^\times)^{2g} \text{ acts on } \check{\mathcal{M}}^d$$

$$\hat{\mathcal{M}}_B^d := \check{\mathcal{M}}_B^d / \Gamma \cong \mathcal{M}_B^d / (\mathbb{C}^\times)^{2g} \text{ is an affine orbifold}$$

Theorem (Non-Abelian Hodge Theorem; Simpson, Corlette)

$$\hat{\mathcal{M}}_{\text{Dol}}^d \stackrel{\text{diff}}{\cong} \hat{\mathcal{M}}_{\text{DR}} \stackrel{\text{RH}}{\cong} \hat{\mathcal{M}}_B$$

- RH is complex analytic \cong ; so SYZ satisfied by $\check{\mathcal{M}}_B^d$ and $\hat{\mathcal{M}}_B^d$

Conjecture (Hausel-Villegas, 2004)

$$(d, n) = (e, n) = 1 \quad E(\check{\mathcal{M}}_B^d; u, v) = E_{st}^{\hat{B}^d}(\mathcal{M}_B^e; u, v)$$

Arithmetic technique to calculate E -polynomials

- E -polynomial of a complex variety X :
$$E(X; u, v) = \sum_{i,p,q} (-1)^i h^{p,q}(Gr_k^W H_c^i(X)) u^p v^q$$
where $W_0 \subseteq W_1 \subseteq \dots \subseteq W_i \subseteq \dots \subseteq W_{2k} = H_c^k(X)$ is the weight filtration.
- \hat{M}_B have a Hodge-Tate type MHS i.e. $h^{p,q} \neq 0$ unless $p = q$
$$E(X; u, v) = E(X, uv) := \sum_{i,k} (-1)^i \dim(Gr_k^W H_c^i(X)) (uv)^k,$$
but the MHS is not pure, i.e. $k \neq i$ when $h^{(k/2,k/2)} \neq 0$.
- X/\mathbb{Z} has *polynomial-count*, if
$$E(q) = |X(\mathbb{F}_q)| \in \mathbb{Q}[q]$$
 is polynomial in q .

Theorem (Katz, 2006)

When X/\mathbb{Z} has *polynomial-count* $E(X/\mathbb{C}, q) = |X(\mathbb{F}_q)|$

- $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ over \mathbb{Z} as the subscheme $\{xy = 1\}$ of \mathbb{A}^2 . Then
$$E(\mathbb{C}^*; q) = |(\mathbb{F}_q^*)| = q - 1$$
- since $H_c^2(\mathbb{C}^*)$ has weight q and $H_c^1(\mathbb{C}^*)$ has weight $1 \rightsquigarrow$ checks with Katz

- for any finite group G , [Frobenius 1896], ..., ..., TQFT [Freed–Quinn 1993] \rightsquigarrow

$$\left| \left\{ a_1, b_1, \dots, a_g, b_g \in G \mid \prod [a_i, b_i] = z \right\} \right| = \sum_{\chi \in \text{Irr}(G)} \frac{|G|^{2g-1}}{\chi(1)^{2g-1}} \chi(z)$$

- when $\zeta_n \in \mathbb{F}_q^*$, i.e. $n \mid q-1$, we get

$$E(\mathcal{M}_B; q) \stackrel{\text{Katz}}{=} |\mathcal{M}_B^d(\mathbb{F}_q)| = (q-1) \sum_{\chi \in \text{Irr}(\text{GL}_n(\mathbb{F}_q))} \frac{|\text{GL}_n(\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-2}} \cdot \frac{\chi(\zeta_n^d \cdot I)}{\chi(1)}$$

$\text{Irr}(\text{GL}_n(\mathbb{F}_q))$ described combinatorially by [Green, 1955] \rightsquigarrow
formula for $E(\mathcal{M}_B; q)$ [Hausel–Villegas, 2008]

- when $n \mid q-1$

$$E(\check{\mathcal{M}}_B; q) \stackrel{\text{Katz}}{=} |\check{\mathcal{M}}_B^d(\mathbb{F}_q)| = \sum_{\chi \in \text{Irr}(\text{SL}_n(\mathbb{F}_q))} \frac{|\text{SL}_n(\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-2}} \cdot \frac{\chi(\zeta_n^d \cdot I)}{\chi(1)}$$

$\text{Irr}(\text{SL}_n(\mathbb{F}_q))$ more difficult; only need value of $\chi(\zeta_n^d \cdot I)$ \rightsquigarrow
Clifford theory \rightsquigarrow calculation of $E(\check{\mathcal{M}}_B; q)$ by [Mereb, 2010]

Character table of $\mathrm{GL}_2(\mathbb{F}_q)$

Table 1: characters of $\mathbf{GL}_2(\mathbb{F}_q)$
 (note that $|\mathbf{GL}_2(\mathbb{F}_q)| = q(q-1)^2(q+1)$)

Classes	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ $a \in \mathbb{F}_q^\times$	$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ $a, b \in \mathbb{F}_q^\times$ $a \neq b$	$\begin{pmatrix} x & 0 \\ 0 & {}^F x \end{pmatrix}$ $x \in \mathbb{F}_{q^2}^\times$ $x \neq {}^F x$	$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ $a \in \mathbb{F}_q^\times$
Number of classes of this type	$q-1$	$\frac{(q-1)(q-2)}{2}$	$\frac{q(q-1)}{2}$	$q-1$
Cardinal of the class	1	$q(q+1)$	$q(q-1)$	q^2-1
$R_{\mathbf{T}}^{\mathbf{G}}(\alpha, \beta)$ $\alpha, \beta \in \mathrm{Irr}(\mathbb{F}_q^\times)$ $\alpha \neq \beta$	$(q+1)\alpha(a)\beta(a)$	$\alpha(a)\beta(b) + \alpha(b)\beta(a)$	0	$\alpha(a)\beta(a)$
$-R_{\mathbf{T}_s}^{\mathbf{G}}(\omega)$ $\omega \in \mathrm{Irr}(\mathbb{F}_{q^2}^\times)$ $\omega \neq \omega^q$	$(q-1)\omega(a)$	0	$-\omega(x) - \omega({}^F x)$	$-\omega(a)$
$\mathrm{Id}_{\mathbf{G}}.(\alpha \circ \det)$ $\alpha \in \mathrm{Irr}(\mathbb{F}_q^\times)$	$\alpha(a^2)$	$\alpha(ab)$	$\alpha(x \cdot {}^F x)$	$\alpha(a^2)$
$\mathrm{St}_{\mathbf{G}}.(\alpha \circ \det)$ $\alpha \in \mathrm{Irr}(\mathbb{F}_q^\times)$	$q\alpha(a^2)$	$\alpha(ab)$	$-\alpha(x \cdot {}^F x)$	0

Character table of $\mathrm{SL}_2(\mathbb{F}_q)$

Table 2: characters of $\mathbf{SL}_2(\mathbb{F}_q)$ for q odd
(note that $|\mathbf{SL}_2(\mathbb{F}_q)| = q(q-1)(q+1)$)

Classes	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ $a \in \{1, -1\}$	$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ $a \in \mathbb{F}_q^\times$ $a \neq \{1, -1\}$	$\begin{pmatrix} x & 0 \\ 0 & {}^F x \end{pmatrix}$ $x, {}^F x \neq 1$ $x \neq {}^F x$	$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ $a \in \{1, -1\}$, $b \in \{1, x\}$ with $x \in \mathbb{F}_q^\times - (\mathbb{F}_q^\times)^2$
Number of classes of this type	2	$(q-3)/2$	$(q-1)/2$	4
Cardinal of the class	1	$q(q+1)$	$q(q-1)$	$(q^2-1)/2$
$R_T^G(\alpha)$ $\alpha \in \mathrm{Irr}(\mathbb{F}_q^\times)$ $\alpha^2 \neq \mathrm{Id}$	$(q+1)\alpha(a)$	$\alpha(a) + \alpha(\frac{1}{a})$	0	$\alpha(a)$
$\chi_{\alpha_0}^\varepsilon$ $\varepsilon \in \{1, -1\}$	$\frac{q+1}{2}\alpha_0(a)$	$\alpha_0(a)$	0	$\frac{\alpha_0(a)}{2}(1 - \varepsilon\alpha_0(ab)\sqrt{\alpha_0(-1)q})$
$-R_T^G(\omega)$ $\omega \in \mathrm{Irr}(\mu_{q+1})$ $\omega^2 \neq \mathrm{Id}$	$(q-1)\omega(a)$	0	$-\omega(x) - \omega({}^F x)$	$-\omega(a)$
$\chi_{\omega_0}^\varepsilon$ $\varepsilon \in \{1, -1\}$	$\frac{q-1}{2}\omega_0(a)$	0	$-\omega_0(x)$	$\frac{\omega_0(a)}{2}(-1 + \varepsilon\alpha_0(ab)\sqrt{\alpha_0(-1)q})$
Id_G	1	1	1	1
St_G	q	1	-1	0

Topological Mirror Test for $n = 2$

- can calculate

$$E_{var}(\check{\mathcal{M}}) = E(\check{\mathcal{M}}) - E(\hat{\mathcal{M}}) = E(\check{\mathcal{M}}) - E(\mathcal{M})/(q-1)^{2g} = \\ (2^{2g} - 1)q^{2g-2} \left(\frac{(q-1)^{2g-2} - (q+1)^{2g-2}}{2} \right) = \\ \sum_{i=1}^{g-1} (2^{2g} - 1) \binom{2g-2}{2i-1} q^{2g-3+2i}$$

- $\check{\mathcal{M}}_\gamma$ can be identified with $(\mathbb{C}^\times)^{2g-2}$ and the Γ -equivariant local system $L_{\beta,\gamma}$ can be explicitly determined \rightsquigarrow

$$E(\check{\mathcal{M}}_\gamma/\Gamma, L_{B,\gamma}) = \frac{(q-1)^{2g-2} - (q+1)^{2g-2}}{2}$$

- $\Rightarrow E(\check{\mathcal{M}}_B) = E_{st}^B(\hat{\mathcal{M}}_B)$ when $n = 2$ due to certain patterns in $Irr(\mathrm{SL}_2(\mathbb{F}_q))$ [Schur, 1907] vs. $Irr(\mathrm{GL}_2(\mathbb{F}_q))$ [Jordan, 1907]
- similar argument works when n is a prime
- for general n one can determine $E(\check{\mathcal{M}}_\gamma/\Gamma, L_{B,\gamma})$ using formulas of Laumon–Ngô and Deligne
- seems to check the Betti-TMS \rightsquigarrow
work in progress with Villegas and Mereb
- $E(\hat{\mathcal{M}}_B; 1/q) = q^d E(\hat{\mathcal{M}}_B; q)$ palindromic \Leftarrow Alvis-Curtis duality in $Irr(G(\mathbb{F}_q))$

Hard Lefschetz for Weight and Perverse Filtrations

- Weight filtration: $W_0 \subset \dots \subset W_i \subset \dots \subset W_{2k} = H^k(X)$
- Alvis-Curtis duality in $R(\mathrm{GL}_n(\mathbb{F}_q))$
 \leadsto Curious Hard Lefschetz Conjecture (theorem for PGL_2):

$$L^l : \underset{x}{\mathrm{Gr}_{d-2l}^W(H^{i-l}(\mathcal{M}_B))} \xrightarrow{\cong} \underset{x \cup \alpha^l}{\mathrm{Gr}_{d+2l}^W H^{i+l}(\mathcal{M}_B)},$$

where $\alpha \in W_4 H^2(\mathcal{M}_B)$

- Perverse filtration: $P_0 \subset \dots \subset P_i \subset \dots \subset P_k(X) \cong H^k(X)$
for $f : X \rightarrow Y$ proper X smooth Y affine
(de Cataldo-Migliorini, 2008):
take $Y_0 \subset \dots \subset Y_i \subset \dots \subset Y_d = Y$
s.t. Y_i generic with $\dim(Y_i) = i$ then

$$P_{k-i-1} H^k(X) = \ker(H^k(X) \rightarrow H^k(f^{-1}(Y_i)))$$

- the Relative Hard Lefschetz Theorem holds:

$$L^l : \underset{x}{\mathrm{Gr}_{d-l}^P(H^*(X))} \xrightarrow{\cong} \underset{x \cup \alpha^l}{\mathrm{Gr}_{d+l}^P H^{*+2l}(X)}$$

where $\alpha \in H^2(X)$ is a relative ample class

$P = W$ conjecture

- recall Hitchin map $\chi : \mathcal{M}_{\text{Dol}} \rightarrow \mathcal{A}$ is proper,
 $(E, \phi) \mapsto \text{charpol}(\phi)$
thus induces perverse filtration on $H^*(\mathcal{M}_{\text{Dol}})$

Conjecture ("P=W", de Cataldo-Hausel-Migliorini 2008)

$P_k(\mathcal{M}_{\text{Dol}}) \cong W_{2k}(\mathcal{M}_{\text{B}})$ under the isomorphism

$H^*(\mathcal{M}_{\text{Dol}}) \cong H^*(\mathcal{M}_{\text{B}})$ from non-Abelian Hodge theory.

Theorem (de Cataldo-Hausel-Migliorini 2009)

$P = W$ when $G = \text{GL}_2, \text{PGL}_2$ or SL_2 .

- Define $PE(\mathcal{M}_{\text{Dol}}; x, y, q) := \sum q^k E(\text{Gr}_k^P(H^*(\mathcal{M}_{\text{Dol}})); x, y)$
- $PE(\mathcal{M}_{\text{Dol}}; x, y, 1) = E(\mathcal{M}_{\text{Dol}}; x, y) = E(\mathcal{M}_{\text{DR}}; x, y)$
- Conjecture $P = W \Rightarrow PE(\mathcal{M}_{\text{Dol}}; 1, 1, q) = E(\mathcal{M}_{\text{B}}; q)$
- $\text{RHL} \rightsquigarrow PE(\mathcal{M}_{\text{Dol}}; x, y, q) = (xyq)^d PE(\mathcal{M}_{\text{Dol}}; x, y; \frac{1}{qxy}) \rightsquigarrow$

Conjecture (Topological Mirror test, TMS)

$$PE_{\text{st}}^{\text{Be}} \left(\mathcal{M}_{\text{Dol}}^d(\text{SL}_n); x, y, q \right) = (xyq)^d PE_{\text{st}}^{\hat{\text{B}}^d} \left(\mathcal{M}_{\text{Dol}}^e(\text{PGL}_n); x, y, \frac{1}{qxy} \right)$$

- The TMS above unifies the previous Dol,DR,B-TMS conjectures (Theorem when $n = 2$)
- Fibrewise Fourier-Mukai transform aka S-duality should identify

$$S : H_p^{r,s}(\mathcal{M}_{\text{Dol}}(\text{SL}_n)) \cong H_{st,d-p}^{r+d/2-p,s+d/2-p}(\mathcal{M}_{\text{Dol}}(\text{PGL}_n))$$

this solves the mirror problem

(Theorem over regular locus of χ)

- (Ngô 2008) proves the fundamental lemma in the Langlands program by proving "geometric stabilisation of the trace formula" which for SL_n and PGL_n can be reformulated to prove TMS over integral spectral curves, which when n is a prime, can be extended to a proof of TMS everywhere.

Some open questions

- Can fibrewise Fourier-Mukai Transform be extended to integral spectral curves? For GL_n the answer is yes by [Arinkin, 2010]
- for reduced, but non-reducible spectral curves? some relevant work by Esteves, López-Martín, . . .
- for non-reduced spectral curves? some recent work by [Drezet, 2009]
- Can the cohomology of the Hitchin fibers computed? for integral (cf. [Ngô, 2008]) reduced but reducible (cf. [Chaudouard-Laumon, 2009]) non-reduced spectral curves?
- Can Gross-Siebert's approach to mirror symmetry (i.e. degenerating the CY's to a reducible one) applied to Hitchin systems? \rightsquigarrow Hitchin systems for singular curves? even only for ordinary double points and for GL_1 ?
- ramifications, other reductive groups?