Kac's conjectures on quiver representations via arithmetic harmonic analysis 3
Topology of Hitchin map and arithmetic of character variety

based on joint work with M. de Cataldo and L. Migliorini

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Quiver varieties, Donaldson-Thomas invariants and instantons,
CIRM Luminy
September 2009

Diffeomorphic spaces in non-Abelian Hodge theory

- C genus g curve; fix group GL_n

$$\mathcal{M}^d_{\mathrm{Dol}} := \left\{ \begin{array}{c} \text{moduli space of semistable rank } n \\ \text{degree } d \text{ } \mathrm{G}\text{-Higgs bundles } (E,\phi) \\ \text{i.e. } E \text{ rank } n \text{ degree d bundle on } C \\ \phi \in H^0(C, ad(E) \otimes K) \text{ Higgs field} \end{array} \right\}$$

$$\mathcal{M}^d_{\mathrm{DR}} := \left\{ \begin{array}{c} \text{moduli space of flat } \mathrm{GL}_n\text{-connections} \\ \text{on } C \setminus \{p\}, \text{ with holonomy } e^{\frac{2\pi i d}{n}} Id \text{ around } p \end{array} \right\}$$

$$\mathcal{M}^d_{\mathrm{B}} := \{A_1, B_1, .., A_g, B_g \in \mathrm{G} | \prod_{i=1}^{g} A_i^{-1} B_i^{-1} A_i B_i = e^{\frac{2\pi i d}{n}} Id\} /\!\!/ \mathrm{G}$$

when $(d, n) = 1$ these are smooth non-compact varieties

Theorem (Non-Abelian Hodge Theorem)

$$\mathcal{M}^d_{\mathrm{Dol}} \overset{diff}{\cong} \mathcal{M}^d_{\mathrm{DR}} \overset{diff}{\cong} \mathcal{M}^d_{\mathrm{B}}$$

Weight filtration on $H^*(\mathcal{M}_{\mathrm{B}})$

- (Deligne 1972) proved the existence of $W_0 \subset \cdots \subset W_i \subset \cdots \subset W_{2k} = H^k(X;\mathbb{Q})$ for any complex algebraic variety X, which is
  - functorial
  - compatible with cup-product
- (Hausel-Villegas 2008) calculates
  $E(\mathcal{M}_{\mathrm{B}};q) = |\mathcal{M}_B(\mathbb{F}_q)| = \sum_{\chi\in Irr(\mathrm{GL}_n(\mathbb{F}_q))} \frac{|\mathrm{GL}_n(\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-1}}\chi(\xi_n)$
- we find $E(\mathcal{M}_{\mathrm{B}};1/q) = q^d E(\mathcal{M}_B;q)$ palindromic by *Alvis-Curtis duality*

  $q^{\frac{n(n-1)}{2}}\chi(1)(1/q) = \chi'(1)(q)$ for dual pair $\chi,\chi' \in Irr(\mathrm{GL}_n(\mathbb{F}_q))$
- $\rightsquigarrow$ Curious Hard Lefschetz Conjecture (theorem when $n=2$):

$$\begin{array}{llcl} L^l: & Gr^W_{d-2l}(H^{i-l}(\mathcal{M}_{\mathrm{B}})) & \to & Gr^W_{d+2l}H^{i+l}(\mathcal{M}_{\mathrm{B}}) \\ & x & \mapsto & x\cup\alpha^l \end{array},$$

  where $\alpha \in W_4H^2(\mathcal{M}_{\mathrm{B}})$
- The implied functional equation on the conjectured $H(\mathcal{M}_{\mathrm{B}};q,t) = (qt)^{d_n}H(\mathcal{M}_{\mathrm{B}};\frac{1}{qt^2},t)$ holds

Perverse filtration

- $f : X \to Y$ a *proper* map between complex algebraic varieties of relative dimension d
- (de Cataldo-Migliorini 2005) introduce *perverse filtration* $\subset P_i \subset P_{i+1} \subset \ldots P_k(X) \cong H^k(X)$ from the study of the Beilinson-Bernstein-Deligne-Gabber decomposition theorem for $Rf_*(\mathbb{Q}_X)$ into perverse sheaves
- the Relative Hard Lefschetz Theorem holds:

$$\begin{array}{cccc} L^l : & Gr^P_{d-l}(H^*(X)) & \longrightarrow & Gr^P_{d+l} H^{*+2l}(X) \\ & x & \mapsto & x \cup \alpha^l \end{array}$$

where $\alpha \in H^2(X)$ is a relative ample class

Main conjecture

- recall Hitchin map $\chi : \begin{array}{ccc} \mathcal{M}_{\mathrm{Dol}} & \to & \mathbb{A} \cong \bigoplus_{i=1}^{n} H^0(C; K^i) \\ (E, \phi) & \mapsto & \mathrm{charpol}(\phi) \end{array}$
- (Hitchin 1987) → completely integrable Hamiltonian system and *proper*

Conjecture ("P=W", de Cataldo-Hausel-Migliorini 2008)

$P_k(\mathcal{M}_{\mathrm{Dol}}) \cong W_{2k}(\mathcal{M}_{\mathrm{B}})$ *under the isomorphism*
$H^*(\mathcal{M}_{\mathrm{Dol}}) \cong H^*(\mathcal{M}_{\mathrm{B}})$ *from non-Abelian Hodge theory*

- recipe (de Cataldo-Migliorini, 2008) for perverse filtration when X smooth and Y affine:
  take $Y_0 \subset \cdots \subset Y_i \subset \ldots Y_d = Y$
  s.t. Y_i generic with $\dim(Y_i) = i$ then

$$P_{k-i-1}H^k(X) = \mathrm{ker}(\mathrm{H^k(X) \to H^k(f^{-1}(Y_i)))}$$

- thus Conjecture ⇒ "topology of Hitchin map reflects the arithmetic of the character variety"

Universal Classes

- now on let $n = 2$, i.e. study $\mathrm{GL}(2)$ Higgs bundles
- $\mathbb{E} \to \mathcal{M}_{\mathrm{Dol}} \times \Sigma$ and $\Phi : \mathbb{E} \to \mathbb{E}K$, universal Higgs bundle $(\mathbb{E}, \Phi)|_{(E,\phi)\times\Sigma} = (E, \phi)$
- $$c_2(End(\mathbb{E})) = 2\alpha[\Sigma]^* + \sum_{i=1}^{2g} 4\psi_i e_i - \beta$$
  for some $\alpha \in H^2(\mathcal{M})$, $\psi_i \in H^3(\mathcal{M})$ and $\beta \in H^4(\mathcal{M})$. Generate $H^*(\mathcal{M}_{\mathrm{Dol}}(\mathrm{PGL}_2))$. (Hausel-Thaddeus 2004)
- (Hausel-Villegas 2008) $\Rightarrow \alpha, \psi_i, \beta \in W_4$,
- Conjecture $\Rightarrow \alpha, \psi_i, \beta \in P_2 \Rightarrow$ $\psi_i, \beta \in \ker(\mathrm{H}^*(\mathcal{M}_{\mathrm{Dol}}) \to \mathrm{H}^*(\chi^{-1}(\mathrm{Y}_0)))$
- Yes! was proved by (Thaddeus 1990)
- $\beta \in P_2 H^4(\mathcal{M}_{\mathrm{Dol}})$ would mean $\beta \in \ker(\mathrm{H}^4(\mathcal{M}_{\mathrm{Dol}}) \to \mathrm{H}^4(\chi^{-1}(\mathrm{Y}_1)))$ i.e. β vanishes over a generic curve in $\mathbb{A}$.

Applications of Ngô's support theorem

Theorem (Ngô, 2008)

$\chi^{ell} : \mathcal{M}_{ell} \subset \mathcal{M}_{\mathrm{Dol}} \to \mathbb{A}_{ell} \subset \mathbb{A}$ over points with integral spectral curve.

$$R\chi_*^{ell}\mathbb{Q} \simeq \bigoplus_{i \geq 0} \mathcal{IC}_{\mathbb{A}_{ell}}(L^{\Lambda i})[-i],$$

where $L^{\Lambda i} = R^i\chi_*^{ell}(\mathbb{Q}) = \Lambda^i R^1\chi_*^{ell}(\mathbb{Q})$ on $\mathbb{A}_{reg}$, where spectral curve is smooth.

Applications for $n = 2$:

- $(2-2g)\beta = c_2(\mathcal{M}_{\mathrm{Dol}})$ vanishes on $\mathbb{A}_{reg}$. Ngô's support theorem $\Rightarrow \beta$ vanishes on the generic line $\Rightarrow \beta \in P_2$.
- By computation $\mathcal{IC}_{\mathbb{A}_{ell}}(L^{\Lambda i}) = j_*^{reg}(L^{\Lambda i})$, i.e. no higher cohomology sheaves $\Rightarrow$ perverse filtration on $H^*(\mathcal{M}_{\mathrm{Dol}}^{ell})$ is compatible with cup-product
  $\Rightarrow P \subset W$ on $H^*(\mathcal{M}_{\mathrm{Dol}})$ (almost)
  $\Rightarrow$ CHL (Thm for $n = 2$) and RHL imply $P = W$ (almost)

$P = W$ when $n = 2$

- One more ingredient needed:
  the intersection form $H_c^*(\mathcal{M}_{\mathrm{Dol}}) \to H^*(\mathcal{M}_{\mathrm{Dol}})$ is trivial when $n = 2$ by (Hausel, 1998)

Theorem (de Cataldo, Hausel, Migliorini 2009)

$P_k(H^*(\mathcal{M}_{\mathrm{Dol}})) \cong W_{2k}(H^*(\mathcal{M}_{\mathrm{B}}))$ *when* $n = 2$.

- Thus in particular $\beta^i \in P_{2i}(H^{4i}(\mathcal{M}_{\mathrm{Dol}}))$
  i.e. vanishes over a generic $2i - 1$ dimensional subvariety in $\mathbb{A}$
- the pure subring $\langle 1, \beta, \dots, \beta^{g-1} \rangle$ is dual by RHL with the g-dimensional $H^{mid}(\mathcal{M}_{\mathrm{Dol}})$
- thus dim $P_{mid/2-2i}/P_{mid/2-2i-1}H^{mid}(\mathcal{M}_{\mathrm{Dol}}) = 1$ for $i = 0, 1, \dots, g - 1$ and 0 otherwise
- consequently $\sum_i q^i$ dim $P_{mid/2-i}H^{mid}(\mathcal{M}_{\mathrm{Dol}}) = A_{S_g}(2, q)$; where S_g is the g-loop quiver

A-polynomial and perverse filtration

- C genus g Riemann surface with punctures $a_1, \ldots, a_k \in C$
- $\mu = (\mu^1, \ldots, \mu^k) \in \mathcal{P}(n)^{\{1..k\}}$
- $\mathcal{M}^{\mu}_{\mathrm{Dol}}$ moduli space of stable parabolic Higgs bundles with generic weights at the quasi-parabolic structure at a_i of type μ^i
- for every μ and g one can find generic weights $\rightsquigarrow \mathcal{M}^{\mu}_{\mathrm{Dol}}$ is always smooth
- can arrange that $\mathcal{M}^{\mu}_{\mathrm{Dol}} \overset{diff}{\cong} \mathcal{M}^{\mu}_{\mathrm{B}}$

Conjecture

$P_k(\mathcal{M}^{\mu}_{\mathrm{Dol}}) \cong W_{2k}(\mathcal{M}^{\mu}_{\mathrm{B}})$ *under the isomorphism*
$H^*(\mathcal{M}^{\mu}_{\mathrm{Dol}}) \cong H^*(\mathcal{M}^{\mu}_{\mathrm{B}})$

Corollary

$\sum_i q^i \dim P_{mid/2-i} H^{mid}(\mathcal{M}^{\mu}_{\mathrm{Dol}}) = A_{\Gamma_\mu}(\alpha_\mu, q)$ *in particular*
$\dim H^{mid}(\mathcal{M}^{\mu}_{\mathrm{Dol}}) = A_{\Gamma_\mu}(\alpha_\mu, 1)$ *and*
$\dim Im(H^{mid}_c(\mathcal{M}^{\mu}_{\mathrm{Dol}})) = A_{\Gamma_\mu}(\alpha_\mu, 0) = m_{\alpha_\mu}$ *for*
$Im(H^{mid}_c(\mathcal{M}^{\mu}_{\mathrm{Dol}})) \subset H^{mid}(\mathcal{M}^{\mu}_{\mathrm{Dol}})$

Hilbert schemes of points on surfaces

- Let $C = E$ elliptic curve, $k = 1$ and $\mu = (\mu^1)$ with $\mu^1 = (n-1, 1)$
- Then one can show that $H^*(\mathcal{M}_{\mathrm{B}}^{\mu}) \cong H^*((\mathbb{C}^\times \times \mathbb{C}^\times)^{[n]})$ preserving the weight filtration
- One can also show that $\mathcal{M}_{\mathrm{Dol}}^{\mu} \cong (T^*E)^{[n]}$ and the Hitchin map is just $(T^*E)^{[n]} \to (T^*E)^{(n)} \to \mathbb{C}^{(n)}$

Theorem (de Cataldo, Hausel, Migliorini 2009)

$P_k(H^*((T^*E)^{[n]})) \cong W_{2k}(H^*((\mathbb{C}^\times \times \mathbb{C}^\times)^{[n]}))$ *under the canonical isomorphism* $H^*((T^*E)^{[n]})) \cong H^*((\mathbb{C}^\times \times \mathbb{C}^\times)^{[n]})$

- in this case μ is indivisible and the quiver variety $\mathcal{M}_{\alpha_\mu} \cong (\mathbb{C}^2)^{[n]}$
- $A_{\Gamma_\mu}(\alpha_\mu, q) = q^n P((\mathbb{C}^2)^{[n]}, 1/\sqrt{q})$ by Crawley-Boevey-Van den Bergh
- so in this case

$$\sum_i q^i \dim P_{mid/2-i} H^{2n}((T^*E)^{[n]}) = q^n P((\mathbb{C}^2)^{[n]}, 1/\sqrt{q}) = A_{\Gamma_\mu}(\alpha_\mu, q)$$

Summary and Outlook

- Cohomology of quiver varieties $\overset{\text{Nakajima}}{\rightsquigarrow}$ representation theory of Kac-Moody algebras
- their Poincaré polynomials $\rightsquigarrow$ A-polynomials of quiver representations
- cohomology of character varieties and moduli of parabolic Higgs bundles (with extra filtrations) $\rightsquigarrow$ deformation of cohomology of quiver varieties
- their weight polynomials $\rightsquigarrow$ deformations of A-polynomials given by Macdonald polynomials
- What is the corresponding deformation of the Kac-Moody algebra?
- Can consider SL_n and PGL_n instead of GL_n $\rightsquigarrow$ Hausel-Thaddeus mirror symmetry conjecture & Ngô's proof of the fundamental lemma in the Langlands program $\rightsquigarrow$ p-adic harmonic analysis
- Connection with our arithmetic harmonic analysis?