Kac's conjectures on quiver representations via arithmetic harmonic analysis 2
Cohomology of character and quiver varieties

based on joint work with E. Letellier and F. R-Villegas

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Quiver varieties, Donaldson-Thomas invariants and instantons,
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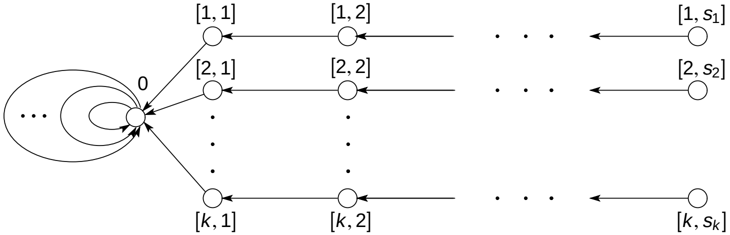
Comet-shaped quiver varieties



- Γ comet-shaped quiver with k legs, and with g loops on the central vertex (star-shaped when $g = 0$)
- $\mu = (\mu^1, \dots, \mu^k) \in \mathcal{P}(n)^{\{1,\dots,k\}}$ be k partititions of n, such that $l(\mu^i)$ is the length of the ith leg
- when μ indivisible i.e. $gcd(\mu^i_j) = 1$ take $\mathcal{C}_j \subset \mathfrak{gl}(n, \mathbb{C})$ are generic semisimple adjoint orbits of type μ^i
- Let α_μ be given on the ith leg by $\alpha_{[i,j]} := n - \sum_{r=1}^{j} \mu^i_r$ and n at vertex 0.

Lemma

When μ is indivisible the symplectic quiver variety

$$\mathcal{M}_{\alpha_\mu} \cong \{(A_1, B_1, \dots, A_g, B_g, C_1, \dots, C_k)\ C_i \in \mathcal{C}_i |$$

$$A_1B_1 - B_1A_1 + \cdots + A_gB_g - B_gA_g + C_1 + \cdots + C_k = 0\} // \mathrm{GL}_n(\mathbb{C})$$

and is smooth.

Character varieties

- C genus g Riemann surface, punctures $a_1, \ldots, a_k \in C$
- $\mu \in \mathcal{P}(n)^{\{1,\ldots,k\}}$ k partitions at the punctures;
  $\tilde{C}_i \subset \mathrm{GL}(n, \mathbb{C})$ semisimple conjugacy classes of type $\mu^i \in \mathcal{P}(n)$
- $$\mathcal{M}_{\mathrm{B}}^{\mu} = \{(A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_k)\, C_i \in \tilde{C}_i | \\ A_1^{-1} B_1^{-1} A_1 B_1 \ldots A_g^{-1} B_g^{-1} A_g B_g C_1 \ldots C_k = Id\} // \mathrm{GL}_n(\mathbb{C})$$
- Taking generic eigenvalues for $\tilde{C}_i$ the character variety can always be made smooth with a given g and μ
- For example, when $k = 1$ and $\mu^1 = (n)$, the generic character variety is smooth:

$$\mathcal{M}_{\mathrm{B}}^{\mu} = \{(A_1, B_1, \ldots, A_g, B_g) | \\ A_1^{-1} B_1^{-1} A_1 B_1 \ldots A_g^{-1} B_g^{-1} A_g B_g = \zeta_n Id\} // \mathrm{GL}_n(\mathbb{C}),$$

where ζ_n is a primitive nth root of unity.

Harmonic analysis on finite groups

- Γ finite group; the *convolution* of $f_1, \dots, f_k : \Gamma \to \mathbb{C}$ is defined:

$$f_1 \star f_2 \star \cdots \star f_k(h) = \sum_{\substack{g_1, g_2, \dots, g_k \in \Gamma \\ g_1 g_2 \dots g_k = h}} f_1(g_1) f_2(g_2) \dots f_k(g_k)$$

- Fourier transform:

$$f : \mathcal{C}(\Gamma) \to \mathbb{C} \rightsquigarrow \hat{f} : \mathrm{Irr}(\Gamma) \to \mathbb{C} \qquad \hat{f} : \mathrm{Irr}(\Gamma) \to \mathbb{C} \rightsquigarrow \hat{\hat{f}} : \mathcal{C}(\Gamma) \to \mathbb{C}$$

$$\hat{f}(\mathcal{X}) = \sum_{C \in \mathcal{C}(\Gamma)} \frac{\mathcal{X}(C) f(C) |C|}{\mathcal{X}(1)} \qquad \hat{\hat{f}}(C) = \sum_{\chi \in \mathrm{Irr}(\Gamma)} \chi(C) \hat{f}(\chi) \chi(1)$$

- Fourier inversion formula: $\hat{\hat{f}}(h) = |\Gamma| f(h^{-1})$

  Fourier of convolution: $\widehat{f_1 \star f_2} = \hat{f}_1 \cdot \hat{f}_2$

- $\#\{a_1 \in C_1, \dots, a_k \in C_k | a_1 a_2 \cdots a_k = 1\} = 1_{C_1} \star \cdots \star 1_{C_k}(1) =$

  $\frac{1}{|\Gamma|} \widehat{\hat{1}_{C_1} \cdots \hat{1}_{C_k}}(1) = \sum_{\chi \in \mathrm{Irr}(\Gamma)} \frac{\chi(1)^2}{|\Gamma|} \prod_{i=1}^{k} \frac{\chi(C_i)|C_i|}{\chi(1)}$

- formula due to (Frobenius 1896)

  $k = 2$ is Frobenius orthogonality

$|GL_3(\mathbb{F}_q)| = (q^3-1)(q^3-q)(q^3-q^2) = q^3(q-1)^3(q+1)(q^2+q+1)$

Tables des caractères de $GL_3(\mathbb{F}_q)$

$|C_G\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}| = q^3(q-1)^2$

$|C_G\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}^F| = q^2(q-1)$

$|Cl_G\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}| = (q^3-1)(q+1)$

$|Cl_{G^F}\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}| = (q^3-1)(q^3-q)$

$|T^F| = (q-1)^3$ | $|W(T)^F| = 6$

$|T_s^F| = (q^2-1)(q-1)$ | $|W(T_s)^F| = 2$

$|T_\sigma^F| = q^3-1$ | $|W(T_\sigma)^F| = 3$

		$\begin{pmatrix}a&0&0\\0&a&0\\0&0&a\end{pmatrix}$ $a\in\mathbb{F}_q^\times$	$\begin{pmatrix}a&0&0\\0&a&0\\0&0&b\end{pmatrix}$ $a\neq b,\ a,b\in\mathbb{F}_q^\times$	$\begin{pmatrix}a&0&0\\0&b&0\\0&0&c\end{pmatrix}$ a,b,c 2 à 2 $\neq$ $\in\mathbb{F}_q^\times$	$\begin{pmatrix}t_1^q&0&0\\0&t_1&0\\0&0&t_2\end{pmatrix}$ $t_1\in\mathbb{F}_{q^2}^\times-\mathbb{F}_q$, $t_2\in\mathbb{F}_q^\times$	$\begin{pmatrix}t_1&0&0\\0&t_1^q&0\\0&0&t_1^{q^2}\end{pmatrix}$ $t_1\in\mathbb{F}_{q^3}^\times-\mathbb{F}_{q^2}$	$\begin{pmatrix}a&0&0\\0&a&1\\0&0&a\end{pmatrix}$ $a\in\mathbb{F}_q^\times$	$\begin{pmatrix}a&1&0\\0&a&1\\0&0&a\end{pmatrix}$ $a\in\mathbb{F}_q^\times$	$\begin{pmatrix}a&0&0\\0&b&1\\0&0&b\end{pmatrix}$ $a\neq b\in\mathbb{F}_q^\times$
	Nombre de classes de ce type	$q-1$	$(q-1)(q-2)$	$\frac{(q-1)(q-2)(q-3)}{6}$	$\frac{q(q-1)(q-1)}{6}$	$\frac{q(q^2-q)}{6}$	$q-1$	$q-1$	$(q-1)(q-2)$
	Cardinal des classes	1	$q^2(q^2+q+1)$	$q^3(q+1)(q^2+q+1)$	$q^2(q^2+q+1)(q^2-1)$	$(q^3-q)(q^3-q^2)$	$(q^3-1)(q+1)$	$(q^3-1)(q^3-q)$	$q^2(q^3-1)(q+1)$
6	$R_T^G(\alpha,\beta,\gamma)$ $\alpha,\beta,\gamma\in Irr(\mathbb{F}_q^\times)$ (α,β,γ) 2 à 2 $\neq$	$(q+1)(q^2+q+1)\times\alpha(a)\beta(a)\gamma(a)$	$(q+1)(\alpha(a)\beta(a)\gamma(b)+\alpha(b)\beta(a)\gamma(a)+\alpha(a)\beta(b)\gamma(a))$	$\sum_{\sigma\in S_3}\alpha(\sigma a)\beta(\sigma b)\gamma(\sigma c)$	0	0	$(1+2q)\times\alpha(a)\beta(a)\gamma(a)$	$\alpha(a)\beta(a)\gamma(a)$	$\alpha(a)\beta(b)\gamma(b)+\alpha(b)\beta(a)\gamma(b)+\alpha(b)\beta(b)\gamma(a)$
1	$Id_{G^F}.(\alpha\circ\det)$ $\alpha\in Irr(\mathbb{F}_q^\times)$	$\alpha(a^3)$	$\alpha(a^2b)$	$\alpha(abc)$	$\alpha(t_1{}^F t_1 t_2)$	$\alpha(t_1{}^F t_1{}^{F^2} t_1)$	$\alpha(a^3)$	$\alpha(a^3)$	$\alpha(ab^2)$
3	$St_G.(\alpha\circ\det)$ $\alpha\in Irr(\mathbb{F}_q^\times)$	$q^3\alpha(a^3)$	$q\alpha(a^2b)$	$\alpha(abc)$	$-\alpha(t_1{}^F t_1 t_2)$	$\alpha(t_1{}^F t_1{}^{F^2} t_1)$	0	0	0
	$R_{T_\sigma}^G(w),\ w\neq w^q$ $w\in Irr(\mathbb{F}_{q^3}^\times)$	$(q^2-1)(q-1)w(a)$	0	0	0	$w(t_1)+w(t_1^q)+w(t_1^{q^2})$	$(1-q)w(a)$	$w(a)$	0
7	$-R_{T_s}^G(w,\alpha)$ $w\in Irr(\mathbb{F}_{q^2}^\times),\ \neq w^q$ $\alpha\in Irr(\mathbb{F}_q^\times)$	$(q^3-1)w(a)\alpha(a)$	$(q-1)w(a)\alpha(b)$	0	$-w(t_1)\alpha(t_2)-w(t_1^q)\alpha(t_2)$	0	$-w(a)\alpha(a)$	$-w(a)\alpha(a)$	$-w(b)\alpha(b)$
2	$R_\theta.(\alpha\circ\det)$ $\alpha\in Irr(\mathbb{F}_q^\times)$	$q(q+1)\alpha(a^3)$	$(q+1)\alpha(a^2b)$	$2\alpha(abc)$	0	$-\alpha(t_1{}^F t_1{}^{F^2} t_1)$	$q\alpha(a^3)$	0	$\alpha(ab^2)$
4	$R_{C_G(s)}^G(Id.(\alpha\circ\det,\beta))$ $\alpha\neq\beta\in Irr(\mathbb{F}_q^\times)$	$(q^2+q+1)\alpha(a^2)\beta(a)$	$\alpha(a^2)\beta(b)+(q+1)\times\alpha(a)\alpha(b)\beta(a)$	$\alpha(ab)\beta(c)+\alpha(ac)\beta(b)+\alpha(bc)\beta(a)$	$\alpha(t_1^{q+1})\beta(t_2)$	0	$(q+1)\alpha(a^2)\beta(a)$	$\alpha(a^2)\beta(a)$	$\alpha(ab)\beta(b)+\alpha(b^2)\beta(a)$
5	$R_{C_G(s)}^G(St_{C_G(s)}.(\alpha\circ\det,\beta))$ $\alpha\neq\beta\in Irr(\mathbb{F}_q^\times)$	$q(q^2+q+1)\times\alpha(a^2)\beta(a)$	$q\alpha(a^2)\beta(b)+(q+1)\times\alpha(a)\alpha(b)\beta(a)$	$\alpha(ab)\beta(c)+\alpha(ac)\beta(b)+\alpha(bc)\beta(a)$	$-\alpha(t_1^{q+1})\beta(t_2)$	0	$q.\alpha(a^2)\beta(a)$	0	$\alpha(b^2)\beta(a)$

Example

$n = 3$, $\tilde{C}_i \subset \mathrm{GL}_3(\mathbb{F}_q)$ regular semisimple:

$$\#\{\mathcal{M}_{\mathrm{B}}^{\mu}(\mathbb{F}_q)\} = \sum_{\chi \in \mathrm{Irr}(\mathrm{GL}_3(\mathbb{F}_q))} \frac{\chi(1)^2}{|\mathrm{GL}_3(\mathbb{F}_q)|} \prod_{i=1}^{k} \frac{\chi(\tilde{C}_i)|\tilde{C}_i|}{\chi(1)} =$$

$$= \frac{\left((q+1)\left(q^2+q+1\right)\right)^k}{\left(q^3-1\right)^2\left(q^2-1\right)^2} - \frac{\left(3\,q^2\,(q+1)\right)^k}{q^4\left(q^2-1\right)^2(q-1)^2}$$

$$+ 1/3\,\frac{\left(6\,q^3\right)^k}{q^6\,(q-1)^4} + \frac{\left(2\,q^2\left(q^2+q+1\right)\right)^k}{q^4\left(q^3-1\right)^2(q-1)^2}$$

$$+ \frac{\left(q^3\,(q+1)\left(q^2+q+1\right)\right)^k}{q^6\left(q^3-1\right)^2\left(q^2-1\right)^2} - \frac{\left(3\,q^3\,(q+1)\right)^k}{q^6\left(q^2-1\right)^2(q-1)^2}.$$

$$\overset{Katz}{=} E(\mathcal{M}_{\mathrm{B}}^{\mu}, q)$$

e.g. $k = 3 \rightsquigarrow \#\{\mathcal{M}_{\mathrm{B}}^{\mu}(\mathbb{F}_q)\} = q^2 + 6q + 1 = E(\mathcal{M}_{\mathrm{B}}^{\mu}, q)$

Weight Polynomials

- (Deligne 1971) proved the existence of $W_0 \subset \cdots \subset W_i \subset \cdots \subset W_{2k} = H^k(X;\mathbb{Q})$ for any complex algebraic variety X
- when X is non-singular $W_{k-1} \cap H^k(X;\mathbb{Q}) = 0$
- $H(X;q,t) = \sum \dim(W_i/W_{i-1}(H^k(X)))t^k q^{\frac{i}{2}}$, *weight polynomial*
- $P(X;t) = H(X;1,t)$, *Poincaré polynomial*
- $E(X;q) = q^d H(1/q,-1)$, *E-polynomial* of X.
- weight filtration on $H^*(X,\mathbb{C})$ is *pure* if $\dim(W_i/W_{i-1}(H^k(X))) = 0$ unless $i = k \Leftrightarrow$ $H(X;q,t) = E(X;(-t\sqrt{q})^2) \Rightarrow$ $P(X;t) = H(X;1,t) = E(X;(-t)^2))$; examples: smooth projective varieties, $\mathcal{M}_{\mathrm{Dol}}$, $\mathcal{M}_{\mathrm{DR}}$, $\mathcal{M}_\alpha$
- in general the pure part of $H(X;q,t)$ is $PH(M;t) = \mathrm{Coeff}_{T^0}\left(H(X;qT^2,tT^{-1})\right)$; which, for a smooth X, is always the image of the cohomology of a smooth compactification

Conjecture

When $n = 3$, $\tilde{C}_i$ regular semisimple, $h^{p;k} = h^{d_\mu - p; k + d_\mu - 2p}$

$$H(\mathcal{M}_{\mathrm{B}}^{\mu}, q, t) = \sum h^{p;k} q^p t^k = \frac{\left(\left(qt^2+1\right)\left(q^2t^4+qt^2+1\right)\right)^k}{\left(q^3t^6-1\right)\left(q^3t^4-1\right)\left(q^2t^4-1\right)\left(q^2t^2-1\right)}$$

$$-\frac{\left(3\,q^2t^4\left(qt^2+1\right)\right)^k}{q^4t^8\left(q^2t^4-1\right)\left(q^2t^2-1\right)\left(qt^2-1\right)\left(q-1\right)} + 1/3\,\frac{6^k\left(qt^2\right)^{3\,k}}{q^6t^{12}\left(qt^2-1\right)^2\left(q-1\right)^2} +$$

$$\frac{\left(q^2t^4\left(2\,q^2t^2+qt^2+q+2\right)\right)^k}{q^4t^8\left(q^3t^4-1\right)\left(q^3t^2-1\right)\left(qt^2-1\right)\left(q-1\right)} + \frac{\left(q^3t^6\left(q+1\right)\left(q^2+q+1\right)\right)^k}{q^6t^{12}\left(q^3t^2-1\right)\left(q^3-1\right)\left(q^2t^2-1\right)\left(q^2-1\right)}$$

$$-\frac{\left(3\,q^3t^6\left(q+1\right)\right)^k}{q^6t^{12}\left(q^2t^2-1\right)\left(q^2-1\right)\left(qt^2-1\right)\left(q-1\right)},$$

e.g. $k = 3 \rightsquigarrow H(\mathcal{M}_{\mathrm{B}}^{\mu}; q, t) = 1 + 6qt^2 + q^2t^2$

The Riemann-Hilbert map

- $\mathfrak{g} = \mathfrak{gl}_n$
- $C = \mathbb{P}^1$ with punctures $a_1, \dots, a_k \in \mathbb{P}^1$; $\mu \in \mathcal{P}(n)^{\{1..k\}}$
- $\mathcal{C}_i$ semisimple adjoint orbit in $\mathfrak{g}(\mathbb{C})$ of type μ^i
- $\mathcal{M}_{\alpha_\mu} = \{(A_1, \dots, A_k)\, A_i \in \mathcal{C}_i | A_1 + \dots + A_k = 0\}//\mathrm{GL}_n(\mathbb{C})$, *star-shaped quiver variety*, as symplectic quotient
- $\mathcal{M}_{\alpha_\mu}$ is smooth when $\mathcal{C}_i$ generic
- "$\mathcal{M}_{\alpha_\mu} \subset \mathcal{M}_{\mathrm{DR}}$", a point in $\mathcal{M}_{\alpha_\mu}$ gives the meromorphic flat $GL(n, \mathbb{C})$-connection $\sum A_i \frac{dz}{z-a_i}$ on the trivial bundle on C.
- $\tilde{\mathcal{C}}_i = \exp(2\pi i \mathcal{C}_i) \subset \mathrm{GL}_n(\mathbb{C})$ is the corresponding conjugacy class
- the *Riemann-Hilbert monodromy map*

$$\nu_a : \mathcal{M}_{\alpha_\mu} \to \mathcal{M}_{\mathrm{B}}^{\mu}$$

is given by sending the flat connection to its holonomy.

Purity conjectures

Conjecture

If $g = 0$ and $\gcd(\mu) = 1$ and C_i are generic of type μ, then

$$\nu_a^* : PH^*(\mathcal{M}_{\mathrm{B}}^{\mu}) \xrightarrow{\cong} H^*(\mathcal{M}_{\alpha_\mu})$$

Conjecture

For every g and $\gcd(\mu) = 1$ and C_i are generic of type μ, then

$$PP(\mathcal{M}_{\mathrm{B}}^{\mu}, t) = P(\mathcal{M}_{\alpha_\mu}, t)$$

Conjecture

For every g and μ if $\tilde{C}_i$ are generic of type μ, then

$$q^{d_\mu/2} PP(\mathcal{M}_{\mathrm{B}}^{\mu}, q^{-1/2}) = A_{\Gamma_\mu}(\alpha_\mu, q)$$

- $n = 3$, $k = 3$, C_i regular i.e. $\mu = (1^3, 1^3, 1^3)$ semisimple
- $\mathcal{M}_{\alpha_\mu}$ is E_6 ALE space,
- $\mathcal{M}_{\mathrm{B}}^{\mu} \cong \mathcal{M}_{\mathrm{Dol}}^{\mu}$ elliptic fibration with singular fibre of type $\hat{E}_6$.
- $P(\mathcal{M}_{\alpha_\mu}; t) = 1 + 6t^2$
- $H(\mathcal{M}_{\mathrm{B}}^{\mu}; q, t) = 1 + 6qt^2 + q^2t^2 \Rightarrow$ Conjecture is true here

Fourier transform for $\mathcal{M}_{\alpha_\mu}$

- $\Psi : \mathbb{F}_q \to \mathbb{C}$ non-trivial additive character
- $f : \mathfrak{g}(\mathbb{F}_q) \to \mathbb{C}$ its Fourier transform $\hat{f} : \mathfrak{g}^*(\mathbb{F}_q) \to \mathbb{C}$

$$\hat{f}(Y) := \sum_{X \in \mathfrak{g}(\mathbb{F}_q)} f(X)\Psi(\langle X, Y \rangle).$$

- $1_{C_i} : \mathfrak{g}(\mathbb{F}_q) \to \mathbb{C}$ characteristic function of $C_i \subset \mathfrak{g}(\mathbb{F}_q)$

Theorem

$$|\mathrm{PGL}(\mathrm{n}, \mathbb{F}_\mathrm{q})| \# \{\mathcal{M}_{\alpha_\mu}(\mathbb{F}_\mathrm{q})\} =$$
$$\#\{A_1 \in C_1, \ldots, A_k \in C_k | A_1 + A_2 + \cdots + A_k = 0\} =$$
$$1_{C_1} \star \cdots \star 1_{C_k}(0) = \widehat{\hat{1}_{C_1} \cdots \hat{1}_{C_k}}(0) =$$
$$\frac{1}{|\mathfrak{gl}_n(\mathbb{F}_q)|} \sum_{X \in \mathfrak{g}(\mathbb{F}_q)} \hat{1}_{C_1}(X) \cdot \cdots \cdot \hat{1}_{C_k}(X)$$

To HAUSEL

Emmanuel Letellier:

Character table of $Gl_3(\mathbb{F}_q)$

	$\begin{pmatrix} a&0&0\\0&a&0\\0&0&a\end{pmatrix}$ $a\in\mathbb{F}_q$	$\begin{pmatrix} a&0&0\\0&a&0\\0&0&b\end{pmatrix}$ $a\neq b$, $a,b\in\mathbb{F}_q$	$\begin{pmatrix} a&0&0\\0&b&0\\0&0&c\end{pmatrix}$ $a,b,c\in\mathbb{F}_q$, $a\neq b, a\neq c, b\neq c$	$\begin{pmatrix} t_1^q&0&0\\0&t_1&0\\0&0&t_2\end{pmatrix}$ $t_1\in\mathbb{F}_{q^2}-\mathbb{F}_q$, $t_2\in\mathbb{F}_q$	$\begin{pmatrix} t_1&0&0\\0&t_1^q&0\\0&0&t_1^{q^2}\end{pmatrix}$ $t_1\in\mathbb{F}_{q^3}-\mathbb{F}_{q^2}$	$\begin{pmatrix} a&0&0\\0&a&1\\0&0&a\end{pmatrix}$ $a\in\mathbb{F}_q$	$\begin{pmatrix} a&1&0\\0&a&1\\0&0&a\end{pmatrix}$ $a\in\mathbb{F}_q$	$\begin{pmatrix} a&0&0\\0&b&1\\0&0&b\end{pmatrix}$ $a\neq b$, $b,a\in\mathbb{F}_q$
$\mathcal{F}\begin{pmatrix}\alpha&0&0\\0&\alpha&0\\0&0&\alpha\end{pmatrix}$ $\alpha\in\mathbb{F}_q$	$q^{-\frac{9}{2}}\psi(a\alpha)^3$	$q^{-\frac{9}{2}}\psi(a\alpha)^2\psi(b\alpha)$	$q^{-\frac{9}{2}}\psi(a\alpha+b\alpha+c\alpha)$	$q^{-\frac{9}{2}}\psi(\alpha(t_1+t_1^q))\psi(\alpha t_2)$	$q^{-\frac{9}{2}}\psi(\alpha(t_1+t_1^q+t_1^{q^2}))$	$q^{-\frac{9}{2}}\psi(a\alpha)^3$	$q^{-\frac{9}{2}}\psi(a\alpha)^3$	$q^{-\frac{9}{2}}\psi(a\alpha)\psi(b\alpha)$ [illegible]
$\mathcal{F}\begin{pmatrix}\alpha&0&0\\0&\alpha&0\\0&0&\beta\end{pmatrix}$ $\alpha\neq\beta$, $\alpha,\beta\in\mathbb{F}_q$	$q^{-\frac{5}{2}}(q^2+q+1)\times\psi(a\alpha)^2\psi(b\beta)$	$q^{-\frac{5}{2}}[\psi(\alpha a)^2\psi(\beta b)+(q+1)\psi(\alpha b)\psi(\alpha a)\times\psi(\beta a)]$	$q^{-\frac{5}{2}}[\psi(\alpha a+\alpha b+\beta c)+\psi(\alpha a+\alpha c+\beta b)+\psi(\alpha b+\alpha c+\beta a)]$	$q^{-\frac{5}{2}}\psi(\alpha(t_1+t_1^q))\times\psi(\beta t_2)$	0	$q^{-\frac{5}{2}}(1+q)\psi(\alpha a)^2\psi(a\beta)$	$q^{-\frac{5}{2}}\psi(\alpha a)^2\psi(a\beta)$	$q^{-\frac{5}{2}}[\psi(\alpha b)^2\psi(\beta a)+\psi(\alpha b)\psi(\alpha a)\psi(a\beta)$
$\mathcal{F}\begin{pmatrix}\alpha&0&0\\0&\beta&0\\0&0&\gamma\end{pmatrix}$ $\alpha,\beta,\gamma\in\mathbb{F}_q$, $\alpha\neq\beta, \alpha\neq\gamma, \beta\neq\gamma$	$q^{-\frac{3}{2}}(q+1)(q^2+q+1)\psi(\alpha a)\psi(\beta a)\psi(\gamma a)$	$q^{-\frac{3}{2}}(q+1)\times[\psi(\alpha a)\psi(\beta a)\psi(\gamma b)+\psi(\alpha a)\psi(\beta b)\psi(\gamma a)+\psi(\alpha b)\psi(\beta a)\psi(\gamma a)]$	$q^{-\frac{3}{2}}\sum_{\sigma\in S_3}\psi(\alpha\sigma(a))\psi(\beta\sigma(b))\psi(\gamma\sigma(c))$	0	0	$q^{-\frac{3}{2}}(1+2q)\psi(\alpha a)\psi(\beta a)\times\psi(\gamma a)$	$q^{-\frac{3}{2}}\psi(\alpha a)\psi(\beta a)\times\psi(\gamma a)$	$q^{-\frac{3}{2}}[\psi(\alpha a)\psi(\beta b)\psi($ [illegible] $+\psi(\alpha b)\psi(\beta b)\psi($ [illegible] $+\psi(\alpha b)\psi(\beta a)\psi($ [illegible]
$\mathcal{F}\begin{pmatrix}r_1^q&0&0\\0&r_1&0\\0&0&r_2\end{pmatrix}$ $r_1\in\mathbb{F}_{q^2}-\mathbb{F}_q$, $r_2\in\mathbb{F}_q$	$q^{-\frac{3}{2}}(q^3-1)\psi(a(r_1+r_1^q))\psi(ar_2)$	$q^{-\frac{3}{2}}(q-1)\psi(a(r_1+r_1^q))\psi(br_2)$	0	$-q^{-\frac{3}{2}}[\psi(r_1^qt_1^q+r_1t_1)\psi(r_2t_2)+\psi(r_2t_2)\psi(r_1t_1^q+r_1^qt_1)]$	0	$-q^{-\frac{3}{2}}\psi(ar_2)\times\psi(a(r_1+r_1^q))$	$-q^{-\frac{3}{2}}\psi(ar_2)\times\psi(a(r_1+r_1^q))$	$-q^{-\frac{3}{2}}\psi(br_2)\times\psi(a(r_1+r_1^q))$
$\mathcal{F}\begin{pmatrix}r&0&0\\0&r^q&0\\0&0&r^{q^2}\end{pmatrix}$ $r\in\mathbb{F}_{q^3}-\mathbb{F}_{q^2}$	$q^{-\frac{3}{2}}(q^2-1)(q-1)\psi(a(r+r^q+r^{q^2}))$	0	0	0	$\psi(rt_1+(rt_1)^q+(rt_1)^{q^2})+\psi(rt_1^q+r^qt_1^{q^2}+r^{q^2}t_1)+\psi(rt_1^{q^2}+r^qt_1+r^{q^2}t_1^q)$	$q^{-\frac{3}{2}}(1-q)\psi(a(r+r^q+r^{q^2}))$	$q^{-\frac{3}{2}}\psi(a(r+r^q+r^{q^2}))$	0
$\mathcal{F}\begin{pmatrix}\alpha&0&0\\0&\alpha&1\\0&0&\alpha\end{pmatrix}$ $\alpha\in\mathbb{F}_q$	$q^{-\frac{3}{2}}(q^3-1)(q+1)\times\psi(a\alpha)^3$	$q^{-9/2}(q^2-1)(q+1)\psi(a\alpha)^2\psi(\alpha b)$	$q^{-9/2}(2q+1)(q-1)\psi(\alpha a)\psi(\alpha b)\psi(\alpha c)$	$-q^{-9/2}(q+1)\psi(\alpha(t_1+t_1^q))\psi(\alpha t_2)$	$-q^{-9/2}(q^2+q+1)\psi(\alpha(t_1+t_1^q+t_1^{q^2}))$	$q^{-3/2}(q^3-q-1)\psi(\alpha a)^3$	$-q^{-7/2}(q+1)\psi(\alpha a)^3$	$q^{-9/2}(q^2-q-1)\psi(\alpha a)\psi(\alpha b)^2$
$\mathcal{F}\begin{pmatrix}\alpha&1&0\\0&\alpha&1\\0&0&\alpha\end{pmatrix}$ $\alpha\in\mathbb{F}_q$	$q^{-\frac{7}{2}}(q^3-1)(q^2-1)\times\psi(a\alpha)^3$	$q^{-7/2}(q-1)^2(q+1)\psi(\alpha b)\psi(\alpha a)^2$	$q^{-7/2}(q-1)^2\psi(\alpha a)\psi(\alpha b)\psi(\alpha c)$	$-q^{-7/2}(q^2-1)\psi(\alpha(t_1+t_1^q))\psi(\alpha t_2)$	$q^{-7/2}(q^2+q+1)\psi(\alpha(t_1+t_1^q+t_1^{q^2}))$	$q^{-7/2}(1-q^2)\psi(\alpha a)^3$	$q^{-7/2}\psi(\alpha a)^3$	$q^{-7/2}(1-q)\psi(\alpha a)\psi(\alpha b)^2$
$\mathcal{F}\begin{pmatrix}\alpha&0&0\\0&\beta&1\\0&0&\beta\end{pmatrix}$ $\alpha\neq\beta$, $\alpha,\beta\in\mathbb{F}_q$	$q^{-5/2}(q^4+q^3-q-1)\psi(\alpha a)\psi(\beta a)^2$	$q^{-\frac{5}{2}}(q^2-1)\times[\psi(\alpha b)\psi(\beta a)^2+\psi(\alpha a)\psi(\beta a)\psi(\beta b)]$	$q^{-\frac{5}{2}}(q-1)[\psi(\alpha a+\beta b+\beta c)+\psi(\alpha b+\beta c+\beta a)+\psi(\alpha c+\beta a+\beta b)]$	$-q^{-\frac{5}{2}}(q+1)\psi(\beta(t_1+t_1^q))\psi(\alpha t_2)$	0	$q^{-5/2}(q^2-q-1)\psi(\beta a)^2\psi(\alpha a)$	$-q^{-5/2}\psi(\beta a)^2\psi(\alpha a)$	$-q^{-5/2}[\psi(\beta b)^2\psi(\alpha$ [illegible] $(1-q)\psi(\alpha b)\psi(\beta a)\psi$ [illegible]

Theorem

Letellier's character table for $\mathfrak{gl}_3(\mathbb{F}_q)$ implies that when $n = 3$ and all adjoint orbits C_i are regular semi-simple

$$P(\mathcal{M}_{\alpha_\mu};t) = \frac{\left(\left(t^2+1\right)\left(t^4+t^2+1\right)\right)^k}{\left(t^6-1\right)\left(t^4-1\right)} - \frac{\left(3\,t^4\left(t^2+1\right)\right)^k}{t^8\left(t^4-1\right)\left(t^2-1\right)} + 1/3\,\frac{6^k\left(t^2\right)^{3\,k}}{t^{12}\left(t^2-1\right)^2} - \frac{\left(t^4\left(t^2+2\right)\right)^k}{t^8\left(t^2-1\right)} + t^{6k-12} + \frac{\left(3\,t^6\right)^k}{t^{12}\left(t^2-1\right)}$$

which agrees with the pure part of the conjectured mixed Hodge polynomial of the corresponding $\mathcal{M}_{\mathrm{B}}^{\mu}$.

Master Conjecture

Conjecture (Hausel-Letellier-Villegas, 2008)

$\mu = (\mu^i)_{i=1}^k \in \mathcal{P}(n)^{\{1,\ldots,k\}}$ *type of the conjugacy classes* $(C_i)_{i=1}^k$

$$H(\mathcal{M}_B^\mu; q, t) = \sum_{p,k} h^{p;k}(\mathcal{M}_B^\mu) q^p t^k = (t\sqrt{q})^{d_\mu}(q-1)(1-\tfrac{1}{qt^2}) \cdot \\ \cdot \left\langle \mathrm{Log}\left(\sum_{\lambda\in\mathcal{P}}\left(\prod_{i=1}^k \tilde{H}_\lambda(\mathbf{x}_i; q, \tfrac{1}{qt^2})\right)\mathcal{H}_\lambda(q, \tfrac{1}{qt^2})\right), h_\mu \right\rangle ,$$

where $\tilde{H}_\lambda(\mathbf{x}_i; q, \frac{1}{qt^2})$ *are the Macdonald symmetric functions.*

Theorem (Hausel-Letellier-Villegas, 2008)

- *The Master Conjecture is true when specialized to* $t = -1$ *giving a formula for* $E(\mathcal{M}_B^\mu; q) = \#\{\mathcal{M}_B^\mu(\mathbb{F}_q)\}$.
- *the pure part of the Master Conjecture gives the Poincaré polynomial of the quiver variety* $\mathcal{M}_{\alpha_\mu}$, *when* μ *is indivisible, and* $A_{\Gamma_\mu}(\alpha_\mu, q)$ *in general.*
- *When* $k = 2$ *the Master Conjecture is true and reduces to the Cauchy identity for Macdonald polynomials; thus it is a deformation of Frobenius' orthogonality for* $\mathrm{GL}_n(\mathbb{F}_q)$.