Kac's conjectures on quiver representations via arithmetic harmonic analysis

Tamás Hausel

Royal Society URF at University of Oxford
http://www.maths.ox.ac.uk/~hausel/talks.html

Quiver varieties, Donaldson-Thomas invariants and instantons,
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Plan for three lectures

Lecture 1: Representations of quivers; Kac's conjectures; Betti numbers of Nakajima's quiver varieties; proof of Kac's conjecture 1

Lecture 2: Cohomology of character and quiver varieties; attack on Kac Conjecture 2

Lecture 3: Topology of Hitchin map and arithmetic of character variety; another attack on Kac Conjecture 2

Quivers and their representations

- a *quiver* Γ is an oriented and connected graph with vertices $I = (1, \ldots, n)$ and arrows or oriented edges $E \subset I \times I$, (possibly multiple edges and edge-loops)
- denote $a = (t(a), h(a)) \in E$ the *tail* and *head* of the arrow a
- $\mathbb{K}$ field; (either $\mathbb{C}$ or $\mathbb{F}_q$)
- a *representation* ρ of Γ is a collection of finite dimensional $\mathbb{K}$-vector spaces $\{V_i\}_{i \in I}$ and homomorphisms $\rho_a \in \mathrm{Hom}(V_{t(a)}, V_{h(a)})$ for every $a \in E$
- $\dim \rho = (\dim V_1, \ldots, \dim V_n) \in \mathbb{N}^I$ is the *dimension* of ρ
- Example: Let S_g be the quiver on one vertex and g loops. Classifying representations of S_g of dimension (d) is classifying the isomorphism classes of g tuples of $d \times d$ matrices. Representations of S_1 (matrices up to conjugation) are classified by Jordan normal form.

The A-polynomial

- $\alpha \in \mathbb{N}^I$ a dimension vector
- a quiver representation is *absolutely indecomposable* if it cannot be written as a direct sum of non-trivial quiver representations over $\overline{\mathbb{K}}$
- $A_\Gamma(\alpha, q) := \left| \left\{ \begin{array}{c} \text{abs. indec. reps of } \Gamma \text{ over } \mathbb{F}_q \text{ of} \\ \text{dimension } \alpha\text{, modulo isomorphism} \end{array} \right\} \right|$

Theorem (Kac, 1982)

$A_\Gamma(\alpha, q) \in \mathbb{Z}[q]$ and is independent of the orientation of Γ.

Kac's conjectures

Conjecture (Kac, 1982)

1. *When* Γ *is loopless, the constant term* $A_\Gamma(\alpha, 0) = m_\alpha$
2. $A_\Gamma(\alpha, q) \in \mathbb{N}[q]$ *, i.e. the coefficients of* $A_\Gamma(\alpha, q)$ *are* ≥ 0.

Both conjectures were known to Kac for finite and affine quivers and for the "polygon"-quiver

Theorem (Crawley-Boevey,Van den Bergh 2004)

Both conjectures hold true for any quiver with α *indivisible; i.e.* $gcd(\alpha(i)) = 1$

Every quiver supports infinitely many divisible dimension vectors $\rightsquigarrow$ both conjectures remained open for any wild quiver

Here we prove Conjecture 1 in Lecture 1 and explain two attacks on Conjecture 2 for comet-shaped quivers in Lectures 2 & 3.

Weight filtration

- X variety defined over $\mathbb{Z}$
- Jordan decomposition of $Frob_q$ on $H_c^k(X(\overline{\mathbb{F}}_q), \mathbb{Q}_\ell) \Rightarrow$ weight filtration $W_l \subset H_c^k$ containing all Jordan blocks of eigenvalue with modulus $q^{i/2}$ $i \leq l$
  ($\Leftarrow$ Weil's Riemann hypothesis $\Leftarrow$ Deligne 1974)
- comparison theorem: $H_c^*(X(\mathbb{C}); \mathbb{C}) \cong H_c^*(X(\overline{\mathbb{F}}_q), \mathbb{Q}_\ell) \otimes \mathbb{C}$
- (Deligne 1971) constructs weight filtration on
  $W_0 \subset \cdots \subset W_i \subset \cdots \subset W_k = H_c^k(X(\mathbb{C}); \mathbb{Q})$ which is functorial
- when $W_{k-1} \cap H_c^k(X; \mathbb{Q}) = 0$
  the weight filtration is *pure*;
  e.g. when X is smooth projective;
  or when $X \subset \overline{X}$, with $\overline{X}$ smooth projective and injects on H_c^*
  e.g. when X is a symplectic quiver variety; a Nakajima quiver variety, $\mathcal{M}_{\mathrm{DR}}$ moduli space of flat connections and $\mathcal{M}_{\mathrm{Dol}}$ the moduli space of Higgs bundles on a Riemann surface
- weight filtration is *not* pure or *mixed* e.g. for $X = \mathrm{GL}_n$ or for $\mathcal{M}_{\mathrm{B}}$ the character variety of representations of the fundamental group of a Riemann surface to GL_n

Arithmetic and topological content of the E-polynomial

- For a complex variety X define *E-polynomial* $E(X;q)=\sum\dim(W_i/W_{i-1}(H_c^k(X)))(-1)^k q^{\frac{i}{2}}$
- additive - if $X_i\subset X$ locally closed s. t. $\dot\cup X_i=X$ then $E(X;q)=\sum E(X_i;q)$
  multiplicative - $F\to E\to B$ locally trivial in the Zariski topology $E(E;q)=E(B;q)E(F;q)$
- when weight filtration is pure $E(X;q)=\sum\dim(H_c^k(X))(-q^{1/2})^k$ is the Poincaré polynomial
- if all eigenvalues λ_i of Frob_q on $H_c^*(X(\overline{\mathbb{F}}_q);\mathbb{Q}_\ell)$ are integer powers of q, then $|X(\mathbb{F}_{q^n})|=\sum\lambda_i^n$ is a *polynomial* in q^n and $=E(X;q)$

Theorem (Katz 2006)

If X is a variety defined over $\mathbb{Z}$ and $\#\{X(\mathbb{F}_q)\}=E(q)$ is a polynomial in q, then $E(M;q)=E(q)$.

e.g. if $E(q)\in\mathbb{Q}[q]\overset{Katz}{\Rightarrow}E(q)\in\mathbb{Z}[q]$ proves Kac's result that $A_\Gamma(\alpha,q)=\#\{\mathcal{Z}(\Gamma,\alpha)(\mathbb{F}_q)\}\in\mathbb{Z}[q]$
$\mathcal{Z}(\Gamma,\alpha)$ is variety parametrizing Γ-indecomposables of dim α

Linear symplectic quotients

- G complex reductive group; $\mathbb{V}$ finite dimensional complex vector space
- assume G acts on $\mathbb{V}$ linearly via the representation $\rho : \mathrm{G} \to \mathrm{GL}(\mathbb{V})$, with derivative $\varrho : \mathfrak{g} \to \mathfrak{gl}(\mathbb{V})$
- symplectic structure on $\mathbb{M} := \mathbb{V} \times \mathbb{V}^*$ given by $\omega((v_1, w_1), (v_2, w_2)) = w_1(v_2) - w_2(v_1)$
- G acts on $\mathbb{V} \times \mathbb{V}^*$symplectically via the representation $\rho \oplus \rho^*$ where $\rho^* : \mathrm{G} \to \mathrm{GL}(\mathbb{V}^*)$ is the dual representation
- this action is Hamiltonian with moment map $\mu : \mathbb{V} \times \mathbb{V}^* \to \mathfrak{g}^*$ $\mu(v, w)(X) = \langle \varrho(X)v, w \rangle$
- for $\xi \in (\mathfrak{g}^*)^{\mathrm{G}}$ we have the *linear symplectic quotient* $\mathbb{M}////_{\xi}\mathrm{G} = \mu^{-1}(\xi)//\mathrm{G}$

Symplectic Quiver varieties

- For a quiver Γ and dimension vector α let $\{V_i\}_{i\in I}$ be a collection of finite dimensional vector spaces of dimension α
- $\mathbb{V}_\alpha = \bigoplus_{a\in E} \mathrm{Hom}(V_{t(a)}, V_{h(a)})$
- $\mathrm{G}_\alpha = \bigtimes_{i\in I} \mathrm{GL}(V_i)/\mathrm{GL}_1$, where $\mathrm{GL}_1 = (\lambda,\ldots,\lambda)_{\lambda\in\mathrm{GL}_1} < \bigtimes_{i\in I} Z(\mathrm{GL}(V_i)) < \bigtimes_{i\in I} \mathrm{GL}(V_i)$
- its Lie algebra $\mathfrak{g}_\alpha = \{X_i \in \mathfrak{gl}(\mathbb{V}_i) | \sum_i \mathrm{tr}(X_i) = 0\} \subset \bigtimes_i \mathfrak{gl}(\mathbb{V}_i)$
- action $\rho : \mathrm{G}_\alpha \to \mathrm{GL}(\mathbb{V}_\alpha)$ from left and right
- for a *generic* $\xi \in (\mathfrak{g}^*_\alpha)^{\mathrm{G}_\alpha}$ define the quiver variety by

$$\mathcal{M}_\alpha = \mathbb{V}_\alpha \times \mathbb{V}^*_\alpha ////_\xi \mathrm{G}_\alpha$$

- if $\alpha \in \mathbb{N}^I$ is indivisible ($\gcd(\alpha) = 1$) then $\mathcal{M}_\alpha$ is non-singular, while if α is divisible ($\gcd(\alpha) > 1$) $\mathcal{M}_\alpha$ has singular points (when non-empty).
- when non-empty $\dim \mathcal{M}_\alpha = 2 - 2(\alpha,\alpha)$
- (Crawley-Boevey, Van den Bergh 2004) when α indivisible $|\mathcal{M}_\alpha(\mathbb{F}_q)| = q^{1-(\alpha,\alpha)} A_\Gamma(\alpha, q)$ & $H^*_c(\mathcal{M}_\alpha;\mathbb{Q})$ is pure $\rightsquigarrow$
  $q^{1-(\alpha,\alpha)} A_\Gamma(\alpha, q) = P_c(\mathcal{M}_\alpha, q^{1/2}) \in \mathbb{N}[q]$
  $\rightsquigarrow$ Kac's Conjecture 2 when α indivisible

Nakajima quiver varieties

- $\mathbf{v}, \mathbf{w} \in \mathbb{N}^I$ and $dim(V_i) = \mathbf{v}_i$ and $\dim(W_i) = \mathbf{w}_i$ then $\mathrm{G}_{\mathbf{v}} = \times_{i \in I} \mathrm{GL}(V_i)$ naturally acts on $\mathbb{V}_{\mathbf{v},\mathbf{w}} = \bigoplus_{(i,j) \in E} \mathrm{Hom}(V_i, V_j) \oplus \bigoplus_{i \in I} \mathrm{Hom}(W_i, V_i)$ the corresponding holomorphic symplectic quotient

$$\mathcal{M}(\mathbf{v}, \mathbf{w}) = \mu^{-1}_{\mathbf{v},\mathbf{w}}(\mathbf{1}_{\mathbf{v}}) /\!/ \mathrm{G}_{\mathbf{v}}$$

  is the affine *Nakajima quiver variety*
- always non-singular of dimension $2d_{\mathbf{v},\mathbf{w}} = 2\left(\sum_{(i,j) \in E} \mathbf{v}_i \mathbf{v}_j + \sum_{i \in I} \mathbf{v}_i(\mathbf{w}_i - \mathbf{v}_i)\right)$
- Crawley-Boevey's trick: to a quiver Γ with two dimension vectors $\mathbf{v}, \mathbf{w} \in \mathbb{N}^I \rightsquigarrow \Gamma_{\mathbf{w}}$ which has $n+1$ vertices $I' = \{1, \ldots, n, *\}$ with the same oriented arrows on $I \subset I'$ and $\mathbf{w}_i$ arrows from $*$ to i. Then one can identify $\mathcal{M}(\mathbf{v}, \mathbf{w}) = \mathcal{M}^{\Gamma_{\mathbf{w}}}_{(\mathbf{v},1)}$, $(\mathbf{v}, 1)$ is clearly indivisible $\Rightarrow$ $P_c(\mathcal{M}_{\mathbf{v},\mathbf{w}}; q^{1/2}) = q^{d_{\mathbf{v},\mathbf{w}}} A_{\Gamma_{\mathbf{w}}}((\mathbf{v}, 1), q)$

Fourier transform on $\mathfrak{g}^*$

- V finite dimensional vector space over $\mathbb{F}_q$
- $\Psi : \mathbb{F}_q \to \mathbb{C}^\times$ non-trivial additive character
- $f : V \to \mathbb{C}$ its Fourier transform $\hat{f} : V^* \to \mathbb{C}$

$$\hat{f}(Y) := \sum_{X \in V} f(X)\Psi(\langle X, Y\rangle).$$

- Fourier inversion formula (FIF): $\hat{\hat{f}}(X) = |V|f(-X)$
- Recall G acts on $\mathbb{V}$, with derivative $\varrho : \mathfrak{g} \to \mathfrak{gl}(\mathbb{V})$, inducing action on $\mathbb{M} := \mathbb{V} \times \mathbb{V}^*$, Hamiltonian with moment map $\mu : \mathbb{M} \to \mathfrak{g}^*$, given by $\mu(v, w)(X) = \langle \varrho(X)v, w\rangle$
- For $\xi \in \mathfrak{g}^*(\mathbb{F}_q)$ the count function of the moment map $\mu : \mathbb{V}(\mathbb{F}_q) \times \mathbb{V}^*(\mathbb{F}_q) \to \mathfrak{g}^*(\mathbb{F}_q)$
  $\#_\mu : \mathfrak{g}^*(\mathbb{F}_q) \to \mathbb{N} \subset \mathbb{C}$
  $\#_\mu(\xi) := \#\{(v, w) \in \mathbb{M}(\mathbb{F}_q) | \mu(v, w) = \xi\} = \sum_{(v,w)\in\mathbb{M}} \delta_{\mu(v,w)}(\xi)$

Proposition (Hausel, 2006)

$$\hat{\#}_\mu(x) = |\mathbb{V}||\ker \varrho(x)| \overset{FIF}{\Longrightarrow} \#_\mu = \frac{|\mathbb{V}|}{|\mathfrak{g}|}\hat{a}_\varrho,$$

where $a_\varrho(x) = |\ker \varrho(x)|$

Betti numbers of Nakajima quiver varieties

Theorem (Hausel 2006)

For any quiver Γ*, and* $\mathbf{w} \in \mathbb{N}^I$ *the Betti numbers of Nakajima quiver varieties are:*

$$\sum_{\mathbf{v}\in\mathbb{N}^I} \sum_i \dim(H_c^{2i}(\mathcal{M}(\mathbf{v},\mathbf{w})))q^{i-d(\mathbf{v},\mathbf{w})} X^{\mathbf{v}} =$$

$$= \frac{\displaystyle\sum_{\mathbf{v}\in\mathbb{N}^I} X^{\mathbf{v}} \sum_{\lambda\in\mathcal{P}(\mathbf{v})} \frac{\left(\prod_{(i,j)\in E} q^{\langle\lambda^i,\lambda^j\rangle}\right)\left(\prod_{i\in I} q^{\langle\lambda^i,(1^{\mathbf{w}_i})\rangle}\right)}{\prod_{i\in I}\left(q^{\langle\lambda^i,\lambda^i\rangle}\prod_k \prod_{j=1}^{m_k(\lambda^i)}(1-q^{-j})\right)}}{\displaystyle\sum_{\mathbf{v}\in\mathbb{N}^I} X^{\mathbf{v}} \sum_{\lambda\in\mathcal{P}(\mathbf{v})} \frac{\prod_{(i,j)\in E} q^{\langle\lambda^i,\lambda^j\rangle}}{\prod_{i\in I}\left(q^{\langle\lambda^i,\lambda^i\rangle}\prod_k \prod_{j=1}^{m_k(\lambda^i)}(1-q^{-j})\right)}},$$

where $2d(\mathbf{v},\mathbf{w}) = 2\sum_{(i,j)\in E}\mathbf{v}_i\mathbf{v}_j + 2\sum_{i\in I}\mathbf{v}_i(\mathbf{w}_i - \mathbf{v}_i)$ *is the dimension of* $\mathcal{M}(\mathbf{v},\mathbf{w})$, $X^{\mathbf{v}} = \prod_{i\in I} T_i^{\mathbf{v}_i}$ *and* $\langle\lambda,\mu\rangle = \sum_{i,j}\min(\lambda_i,\mu_j)$

Weyl-Kac character formula for Nakajima quiver varieties

Theorem (Kac 1974)

Let $L(\mathbf{w})$ be an irreducible representation of $\mathfrak{g}(\Gamma)$ of highest weight $\Lambda \in P$. Let $L(\Lambda) = \oplus_{\alpha\in\mathbb{N}^I} L(\Lambda)_{\Lambda-\alpha}$ denote its weight space decomposition. Then

$$\sum_{\alpha\in\mathbb{N}^I} \dim\left(L(\Lambda)_{\Lambda-\alpha}\right) X^{\alpha} = \frac{\sum_{w\in W} \det(w) X^{\Lambda+\rho-w(\Lambda+\rho)}}{\prod_{\alpha\in\mathbb{N}^I}(1-X^{\alpha})^{m_\alpha}}$$

Theorem (Nakajima 1998)

Fix $\mathbf{w} \in \mathbb{N}^I$ then there is an irreducible representation of the Kac-Moody algebra $\mathfrak{g}(\Gamma)$ of highest weight $\Lambda_{\mathbf{w}}$ on $\bigoplus_{\mathbf{v}\in\mathbb{N}^I} H_c^{2d_{\mathbf{v},\mathbf{w}}}(\mathcal{M}(\mathbf{v},\mathbf{w}))$, in particular

$$\sum_{\mathbf{v}\in\mathbb{N}^I} \dim\left(H^{2d_{\mathbf{v},\mathbf{w}}}(\mathcal{M}(\mathbf{v},\mathbf{w})\right) X^{\mathbf{v}} = \frac{\sum_{w\in W} \det(w) X^{\Lambda_{\mathbf{w}}+\rho-w(\Lambda_{\mathbf{w}}+\rho)}}{\prod_{\alpha\in\mathbb{N}^I}(1-X^{\alpha})^{m_\alpha}}$$

Proof of Kac's Conjecture 1

Weyl-Kac-Nakajima formula + our main formula $\rightsquigarrow$

$$\frac{\sum_{w\in W} \det(w) X^{\Lambda_{\mathbf{w}}+\rho-w(\Lambda_{\mathbf{w}}+\rho)}}{\prod_{\alpha\in\mathbb{N}^I}(1-X^\alpha)^{m_\alpha}} = \left(\frac{\sum_{\mathbf{v}\in\mathbb{N}^I} X^{\mathbf{v}} \sum_{\lambda\in\mathcal{P}(\mathbf{v})} \frac{\left(\prod_{(i,j)\in E} q^{\langle\lambda^i,\lambda^j\rangle}\right)\left(\prod_{i\in I} q^{\langle\lambda^i,(1^{\mathbf{w}_i})\rangle}\right)}{\prod_{i\in I}\left(q^{\langle\lambda^i,\lambda^i\rangle}\prod_k\prod_{j=1}^{m_k(\lambda^i)}(1-q^{-j})\right)}}{\sum_{\mathbf{v}\in\mathbb{N}^I} X^{\mathbf{v}} \sum_{\lambda\in\mathcal{P}(\mathbf{v})} \frac{\prod_{(i,j)\in E} q^{\langle\lambda^i,\lambda^j\rangle}}{\prod_{i\in I}\left(q^{\langle\lambda^i,\lambda^i\rangle}\prod_k\prod_{j=1}^{m_k(\lambda^i)}(1-q^{-j})\right)}} \right)_{q=0}$$

when $\mathbf{w} = m\mathbf{1}$, i.e. $\Lambda_{\mathbf{w}} = m\rho$, $m \to \infty$ and $A_\Gamma(\alpha, q) = \sum_i t_i^\alpha q^i$

$$\prod_{\alpha\in\mathbb{N}^I}(1-X^\alpha)^{m_\alpha} = \left(\sum_{\mathbf{v}\in\mathbb{N}^I} X^{\mathbf{v}} \sum_{\lambda\in\mathcal{P}(\mathbf{v})} \frac{\prod_{(i,j)\in E} q^{\langle\lambda^i,\lambda^j\rangle}}{\prod_{i\in I}\left(q^{\langle\lambda^i,\lambda^i\rangle}\prod_k\prod_{j=1}^{m_k(\lambda^i)}(1-q^{-j})\right)} \right)_{q=0}$$

$$\overset{\text{Hua}}{=} \left(\prod_{\alpha\in\mathbb{N}^n}\prod_{j=0}^{\infty}\prod_{i=0}^{\infty}(1-q^{i+j}X^\alpha)^{t_i^\alpha} \right)_{q=0} = \prod_{\alpha\in\mathbb{N}^n}(1-X^\alpha)^{t_0^\alpha}$$

Theorem (Hausel 2006)

$A_\Gamma(\alpha, 0) = m_\alpha$