Global topology of the Hitchin system

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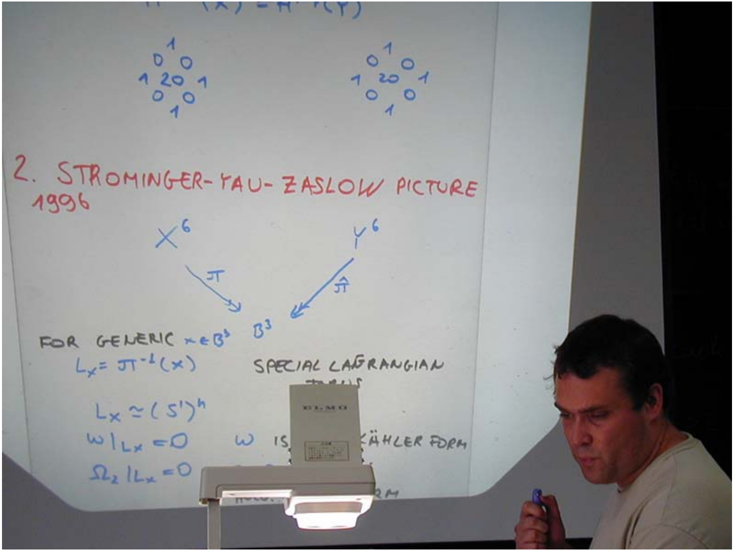

Mirror Symmetry

- phenomenon first arose in various forms in string theory
- mathematical predictions (Candelas-de la Ossa-Green-Parkes 1991)
- mathematically it relates the symplectic geometry of a Calabi-Yau manifold X^d to the complex geometry of its mirror Calabi-Yau Y^d
- first aspect is the *topological mirror test* $h^{p,q}(X) = h^{d-p,q}(Y)$
- compact hyperkähler manifolds satisfy $h^{p,q}(X) = h^{d-p,q}(X)$
- (Kontsevich 1994) suggests *homological mirror symmetry* $\mathcal{D}^b(Fuk(X,\omega)) \cong \mathcal{D}^b(Coh(Y,I))$
- (Strominger-Yau-Zaslow 1996) suggests a geometrical construction how to obtain Y from X
- many predictions of mirror symmetry have been confirmed - no general understanding yet

Fermat quintic X

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

$\hat{X} := X/(\mathbb{Z}_5)^3$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 101 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 101 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

K3 surface X

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$\hat{X}$ mirror K3

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Strominger-Yau-Zaslow

- X CY 3-fold
- Y mirror CY 3-fold
- B is 3-dimensional real manifold - mostly S^3

$$X^6 \xrightarrow{\pi} B^3 \xleftarrow{\hat{\pi}} Y^6$$

- π and $\hat{\pi}$ are special Lagrangian fibrations
- for generic $x \in B^3$
  $L_x = \pi^{-1}(x) \cong T^3$ and $\hat{L}_x = \hat{\pi}^{-1}(x) \cong T^3$ are dual special Lagrangian tori
- generically Y^6 can be thought of as the moduli space of flat $U(1)$ connections on a generic fiber L_x (a.k.a. *D-branes*)

Langlands duality

- the Langlands program aims to describe $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ via representation theory
- G reductive group, ${}^L\mathrm{G}$ its Langlands dual
- e.g ${}^L\mathrm{GL}_n = \mathrm{GL}_n$; ${}^L\mathrm{SL}_n = \mathrm{PGL}_n$, ${}^L\mathrm{PGL}_n = \mathrm{SL}_n$
- [Langlands 1967] conjectures that $\{\text{homs } \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{G}(\mathbb{C})\} \leftrightarrow \{\text{automorphic reps of } {}^L\mathrm{G}(\mathcal{A}_{\mathbb{Q}})\}$
- $\mathrm{G} = \mathrm{GL}_1 \rightsquigarrow$ class field theory
  $\mathrm{G} = \mathrm{GL}_2 \rightsquigarrow$ Shimura-Taniyama-Weil
- function field version: replace $\mathbb{Q}$ with $\mathbb{F}_q(X)$, where $X/\mathbb{F}_q$ is algebraic curve
- [Ngô, 2008] proves fundamental lemma for $\mathbb{F}_q(X) \rightsquigarrow$ FL for $\mathbb{Q}$
- geometric version: replace $\mathbb{F}_q(X)$ with $\mathbb{C}(X)$ for $X/\mathbb{C}$
- [Laumon 1987, Beilinson–Drinfeld 1995]
  Geometric Langlands conjecture
  $\{\mathrm{G}\text{-local systems on } X\} \leftrightarrow \{\text{Hecke eigensheaves on } \mathrm{Bun}_{{}^L\mathrm{G}}(X)\}$
- [Kapustin–Witten 2006] deduces this from reduction of S-duality (electro-magnetic duality) in $N = 4$ SUSY YM in $4d$

Hitchin system

- Hamiltonian system: (X^{2d}, ω) symplectic manifold
  $H : X \to \mathbb{R}$ Hamiltonian function X_H Hamiltonian vector field $(dH = \omega(X_H, .))$
- $f : X \to \mathbb{R}$ is a *first integral* if $X_H f = \omega(X_f, X_H) = 0$
- the Hamiltonian system is *completely integrable* if there is $f = (H = f_1, \ldots, f_d) : X \to \mathbb{R}^d$ generic such that $\omega(X_{f_i}, X_{f_j}) = 0$
- the generic fibre of f has an action of $\mathbb{R}^d = \langle X_{f_1}, \ldots, X_{f_d} \rangle \rightsquigarrow$ when f is proper generic fibre is a torus $(S^1)^d$
- examples include: Euler and Kovalevskaya tops and the spherical pendulum
- algebraic version when replacing $\mathbb{R}$ by $\mathbb{C} \rightsquigarrow$ many examples can be formulated as a version of the *Hitchin system*
- a Hitchin system is associated to a complex curve C and a complex reductive group G
- it arose in the study [Hitchin 1987] of the 2-dimensional reduction of the Yang-Mills equations

Topological mirror tests

- In these lectures we will discuss the mirror symmetry proposal of [Hausel–Thaddeus 2003]:
  "Hitchin systems for Langlands dual groups satisfy Strominger-Yau-Zaslow, so could be considered mirror symmetric; in particular they should satisfy the *topological mirror tests*:"
- $$\mathcal{M}^d_{\mathrm{DR}}(\mathrm{SL}_n) \xrightarrow{\check{\chi}} \mathcal{A}^0 \xleftarrow{\hat{\chi}} \mathcal{M}^e_{\mathrm{DR}}(\mathrm{PGL}_n)$$

Conjecture (Hausel–Thaddeus 2003, "Topological mirror test")

For all $d, e \in \mathbb{Z}$, satisfying $(d, n) = (e, n) = 1$, we have

$$E_{\mathrm{st}}^{B^e}\left(\mathcal{M}^d_{\mathrm{DR}}(\mathrm{SL}_n); x, y\right) = E_{\mathrm{st}}^{\hat{B}^d}\left(\mathcal{M}^e_{\mathrm{DR}}(\mathrm{PGL}_n); x, y\right).$$

The moduli space of vector bundles on a curve - GL_n

- C smooth complex projective curve of genus $g > 1$
- fix integers $n > 0$ and $d \in \mathbb{Z}$ always assume $(d, n) = 1$.
- $\mathcal{N}^d :=$ moduli space of isomorphism classes of semi-stable rank n degree d vector bundles on C
- constructed using geometric invariant theory (GIT) or gauge theory
- vector bundle E is called *semi-stable* (*stable*) if every proper subbundle F satisfies

$$\mu(F) = \frac{\deg(F)}{\mathrm{rk}(F)} \overset{(<)}{\leq} \mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)}$$

- when $(d, n) = 1$ semi-stability $\Leftrightarrow$ stability $\rightsquigarrow$ $\mathcal{N}^d$ is a non-singular projective fine moduli space

SL_n and PGL_n

- $$\begin{array}{rccc} \det: & \mathcal{N}^d & \to & \mathrm{Jac}^d(C) \\ & [E] & \mapsto & \Lambda^n(E) \end{array}$$
- fix $\Lambda \in \mathrm{Jac}^d(C)$ and let $\check{\mathcal{N}}^\Lambda := \det^{-1}(\Lambda) \subset \mathcal{N}^d$ the moduli space of (twisted) SL_n bundles on C
- $\check{\mathcal{N}}^\Lambda$ does not depend on the choice of $\Lambda \in \mathrm{Jac}^d(C)$ just write $\check{\mathcal{N}}^d := \check{\mathcal{N}}^\Lambda$
- when $(d,n) = 1 \rightsquigarrow \check{\mathcal{N}}^d$ is non-singular and projective
- $\mathrm{Pic}^0(C) = \mathrm{Jac}^0(C)$ acts on $\mathcal{N}^d$ via $(L, E) \mapsto L \otimes E$. define

$$\hat{\mathcal{N}}^d := \mathcal{N}^d / \mathrm{Pic}^0(C)$$

  the moduli space of degree d PGL_n bundles on C
- $\Gamma := \mathrm{Pic}^0(C)[n] \cong \mathbb{Z}_n^{2g} \subset \mathrm{Pic}^0(C)$ acts on $\hat{\mathcal{N}}^d$ and clearly $\hat{\mathcal{N}}^d = \check{\mathcal{N}}^d / \Gamma \rightsquigarrow \hat{\mathcal{N}}^d$ is a projective orbifold.

Cohomology of $\hat{\check{\mathcal{N}}}$

- The cohomologies $H^*(\mathcal{N}^d)$, $H^*(\check{\mathcal{N}}^d)$ and $H^*(\hat{\mathcal{N}}^d)$ are well understood.
- [Harder–Narasimhan 1975] obtained recursive formulae for $\#\mathcal{N}(\mathbb{F}_q) \rightsquigarrow$ formula for Betti numbers via the Weil conjectures [Deligne 1974]
- [Atiyah–Bott 1981] gave different gauge-theoretic proof

Theorem (Harder–Narasimhan, 1975)

The finite group Γ acts trivially on $H^*(\check{\mathcal{N}}^d)$. In particular $H^*(\check{\mathcal{N}}^d) \cong H^*(\hat{\mathcal{N}}^d)$.

- proof by showing $\#\check{\mathcal{N}}^d(\mathbb{F}_q) = \#\hat{\mathcal{N}}^d(\mathbb{F}_q)$
- [Hitchin, 1987] $\Rightarrow$ false for moduli space of SL_2 Higgs bundles $\rightsquigarrow$ non-triviality of our topological mirror tests

The Hitchin map - GL_n

- $T^*\mathcal{N}$ is a (non-projective) algebraic symplectic variety
- the ring $\mathbb{C}[T^*\mathcal{N}]$ is known to be finitely-generated
- the affinization of $T^*\mathcal{N}$ gives the GL_n *Hitchin map*.

$$\chi : T^*\mathcal{N} \to \mathcal{A} := \mathrm{Spec}(\mathbb{C}[T^*\mathcal{N}])$$

- deformation theory $\rightsquigarrow$ $T_{[E]}\mathcal{N} = H^1(C, \mathrm{End}(E))$
  Serre duality $\Rightarrow$ $T^*_{[E]}\mathcal{N} = H^0(C, \mathrm{End}(E) \otimes K)$
- $\phi \in H^0(C, \mathrm{End}(E) \otimes K)$ is a *Higgs field*
  locally "a matrix of one-forms on the curve"
- let $(E, \phi) \in T^*\mathcal{N}$ its characteristic polynomial
  $\chi(\phi) = t^n + a_1 t^{n-1} + \cdots + a_n$ where $a_i \in H^0(K^i)$
- $$\begin{array}{rcl} \chi : T^*\mathcal{N} & \to & \mathcal{A} := \bigoplus_{i=1}^n H^0(K^i) \\ (E, \phi) & \mapsto & (a_1, a_2, \ldots, a_n) \end{array}$$
- The affine space $\mathcal{A}$ is called the *Hitchin base*.

Hitchin map for SL_n and PGL_n

- for SL_n
$$T^*_{[E]}\check{\mathcal{N}}^d = H^0(\mathrm{End}_0(E) \otimes K)$$
that is, a covector at E is given by a *trace free* Higgs field.
- the SL_n *Hitchin base* is
$$\check{\mathcal{A}} = \mathcal{A}^0 := \bigoplus_{i=2}^{n} H^0(C, K^i).$$
- the SL_n *Hitchin map*
$$\check{\chi} : T^*\check{\mathcal{N}}^d \to \mathcal{A}^0.$$
- the action of $\Gamma = \mathrm{Pic}^0(C)[n]$ on $T^*\check{\mathcal{N}}$ is along the fibers of $\check{\chi}$
$\Rightarrow \check{\chi}$ descends to the quotient
- the PGL_n *Hitchin map*:
$$\hat{\chi} : (T^*\check{\mathcal{N}})/\Gamma \to \hat{\mathcal{A}} = \mathcal{A}^0.$$

The Hitchin map is an integrable system

- recall that $T^*\mathcal{N}$ is an algebraic symplectic variety
- with canonical Liouville symplectic structure
- as the Hitchin map only depends on the cotangent direction
  $\rightsquigarrow$

Theorem (Hitchin, 1987)

- $\omega(X_{\chi_i}, X_{\chi_j}) = 0$ *for any two* $\chi_i, \chi_j \in \mathbb{C}[T^*\mathcal{N}]$ *coordinate functions.*
- $\dim(\mathcal{A}) = \dim(\mathcal{N}) = \dim(T^*\mathcal{N})/2$
- *generic fibres of* χ *are open subsets of abelian varieties*

$\rightsquigarrow$ χ *is an algebraically completely integrable Hamiltonian system.*

- Need to projectivize χ to complete the generic fibres to abelian varieties (compact tori)

Proper Hitchin map

- $(E, \phi) \in T^*\mathcal{N} \rightsquigarrow E$ is stable; to projectivize χ we need to allow E to become unstable.
- A *Higgs bundle* is a pair (E, ϕ) where E is a vector bundle on C and $\phi \in H^0(C, \mathrm{End}(E) \otimes K)$ is a *Higgs field*.
- a Higgs bundle (E, ϕ) is *(semi-)stable* if for every ϕ-invariant proper subbundle E we have $\mu(F) \overset{(\leq)}{<} \mu(E)$
- $\mathcal{M}^d$ the moduli space of (semi-)stable Higgs bundles, a non-singular quasi-projective and symplectic variety, containing $T^*\mathcal{N} \subset \mathcal{M}^d$ as an open dense subvariety
- extend $\chi : \mathcal{M}^d \to \mathcal{A}$ in the obvious way

Theorem (Hitchin 1987, Nitsure 1991, Faltings 1993)

χ is a proper algebraically completely integrable Hamiltonian system. Its generic fibres are abelian varieties.

- as $\mathrm{codim}(\mathcal{M}^d \setminus T^*\mathcal{N}^d) \geq 2 \Rightarrow \mathbb{C}[\mathcal{M}^d] \cong \mathbb{C}[T^*\mathcal{N}^d] \Rightarrow$ thus

$$\mathcal{A} \overset{\text{by the Theorem}}{\cong} \mathrm{Spec}(\mathbb{C}[\mathcal{M}^d]) \cong \mathrm{Spec}(\mathbb{C}[T^*\mathcal{N}^d])$$

SL_n Hitchin system

- fix $\Lambda \in \mathrm{Jac}^d(C)$
- E vector bundle on C with determinant Λ
- $\phi \in H^0(\mathrm{End}_0(E) \otimes K)$ is trace-free Higgs field
- then (E, ϕ) is an SL_n-*Higgs bundle*
- $\check{\mathcal{M}}^\Lambda \subset \mathcal{M}^d$ moduli space of (semi-)stable SL_n-*Higgs bundles*
- $\check{\mathcal{M}}^\Lambda$ is independent of Λ denote $\check{\mathcal{M}}^d := \check{\mathcal{M}}^\Lambda$
- $\check{\mathcal{M}}^d$ is a non-singular quasi-projective and symplectic variety
- characteristic polynomial of ϕ gives SL_n-*Hitchin system*

$$\check{\chi} : \check{\mathcal{M}}^d \to \mathcal{A}^0 := \oplus_{i=2}^n H^0(C; K^i)$$

- $\check{\chi}$ is proper and a completely integrable system

PGL_n Hitchin system over the same Hitchin base

- $T^* \operatorname{Pic}^0(C) = \operatorname{Pic}^0(C) \times H^0(C, K)$ is a group; it acts on $\mathcal{M}^d$ by $(L, \varphi)(E, \phi) \mapsto (L \otimes E, \varphi + \phi)$
- $\rightsquigarrow$ action of $\Gamma = \operatorname{Pic}^0[n]$ on $\check{\mathcal{M}}^d$
- $\hat{\mathcal{M}}^d = \mathcal{M}^d / T^* \operatorname{Pic}^0(C) \cong \chi^{-1}(\mathcal{A}^0)/\operatorname{Pic}^0(C) \cong \check{\mathcal{M}}/\Gamma$
- $\hat{\mathcal{M}}^d$, the PGL_n *Higgs moduli space*, is an orbifold
- the Γ action is along the fibers of $\check{\chi} \rightsquigarrow \mathrm{PGL}_n$ *Hitchin map*

$$\hat{\chi} : \ \hat{\mathcal{M}}^d = \check{\mathcal{M}}^d/\Gamma \to \mathcal{A}^0$$

- $$\begin{array}{ccc} \check{\mathcal{M}}^d & & \hat{\mathcal{M}}^e \\ & \searrow^{\check{\chi}} \quad {}^{\hat{\chi}}\swarrow & \\ & \mathcal{A}^0 & \end{array}$$

- will show generic fibers are dual Abelian varieties; which are complex Lagrangian due to integrable system
- changing complex structure will lead to special Lagrangian fibrations; and so to SYZ

Spectral curves

- let (E, ϕ) be a Higgs bundle such that $\chi(\phi) = a \in \mathcal{A}$ has the form
$$a = t^n + a_1 t^{n-1} + \cdots + a_n,$$
where $a_i \in H^0(K^i)$.
- What should be the spectrum of the Higgs field ϕ?
- at $p \in C$ the Higgs field $\phi_p : E_p \to E_p \otimes K_p$
- eigenvalue ν_p of ϕ_p satisfies $\exists v \in E_p - 0 : \ \Phi_p(v) = \nu_p v. \rightsquigarrow$ must have $\nu_p \in K_p$
- let X denote the total space of K then $C_a := \cup_{p \in C} \nu_p^i \subset X$, the set of all eigenvalues of the Higgs field $\rightsquigarrow$ *spectral curve*
- scheme structure on C_a?
- tautological section $\lambda \in H^0(X, \pi^* K)$ satisfying $\lambda(x) = x$
- $s_a := \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n \in H^0(X, \pi^* K^n)$
- $C_a := s_a^{-1}(0) \subset X$ spectral curve
$\pi_a : C_a \to C$ spectral cover of degree n
- geography of $\mathcal{A}$: $\mathcal{A}_{\mathrm{reg}} \subset \mathcal{A}_{\mathrm{ell}} \subset \mathcal{A}_{\mathrm{hyp}} \subset \mathcal{A}$
C_a is non-singular, C_a is integral, C_a is reduced, C_a anything
- $0 \in \mathcal{A} \setminus \mathcal{A}_{\mathrm{hyp}} \rightsquigarrow C_0 \subset X$, $red(C_0) = C \rightsquigarrow \mathcal{M}_0$ nilpotent cone

Generic fibres of the Hitchin map

- assume C_a is smooth $\Leftrightarrow a \in \mathcal{A}_{reg}$; $(E,\phi) \in \chi^{-1}(a) =: \mathcal{M}_a$
- if $\nu_p \in C_a \subset X$ then $L_{\nu_p} \subset \pi_a^*(E)$ ν_p-eigenspace in $E_p \rightsquigarrow$ $L := \ker(\lambda Id_E - \pi_a^*(\phi)) \subset \pi_a^*(E)$ subsheaf rank 1 $\rightsquigarrow$ invertible as C_a is smooth such that $\pi_*(L(\Delta)) = E$ (eigenspace decomposition of ϕ)
- starting with a line bundle $M \in \mathrm{Jac}^d(C_a)$ we construct $E = \pi_*(M)$ rank n degree $d' = d - n(n-1)(g-1)$ torsion free $\rightsquigarrow$ locally free and Higgs field $\phi := \pi_*(\lambda) : \pi_*(M) \to \pi_*(M) \otimes K$ pushing forward the tautological map $\lambda : M \to M \otimes \pi^*(K)$
- by definition λ solves the characteristic polynomial a on $C_a \rightsquigarrow$ so will $\phi \rightsquigarrow$ by Cayley-Hamilton $\chi(\phi) = a$
- the spectral curve of a proper Higgs subbundle of $(E,\phi) = (\pi_*(M), \pi_*(\lambda))$ would be a 1-dimensional proper subscheme of $C_a \Rightarrow (E,\phi)$ is stable

Theorem (Hitchin 1987, Beauville-Narasimhan-Ramanan 1989)

For $a \in \mathcal{A}_{reg}$ we have $\mathcal{M}_a^{d'} \cong \mathrm{Jac}^d$.

Generic fibers for SL_n and PGL_n-Hitchin map

- recall (E, ϕ) SL_n-Higgs bundle if $\mathrm{tr}(\phi) = 0$ and $\det(E) = \Lambda \in Jac^{d'}(C)$
- define $\mathrm{Prym}^d(C) \subset \mathrm{Jac}^d(C_a)$ by

$$L \in \mathrm{Prym}^d(C_a) \Leftrightarrow \det \pi_*(L) = \Lambda$$

- if $a \in \mathcal{A}^0_{reg}$ the SL_n-Hitchin fibre satisfies

$$\check{\mathcal{M}}_a := \check{\chi}^{-1}(a) \cong \mathrm{Prym}^d(C_a).$$

- for PGL_n we have $\hat{\mathcal{M}}_a := \hat{\chi}^{-1}(a) \cong \check{\mathcal{M}}_a/\Gamma \cong \mathrm{Prym}^d(C_a)/\Gamma$ makes sense since for $L_\gamma \in \mathrm{Pic}(C)[n]$ we have $\det(\pi_*(\pi^*(L_\gamma) \otimes L)) = \det(L_\gamma \otimes \pi_*(L)) = L_\gamma^n \otimes \det(\pi_* L) = \det(\pi_* L)$.
- alternatively $\hat{\mathcal{M}}_a = \mathcal{M}_a / \mathrm{Pic}^0(C) \cong \mathrm{Jac}^d(C_a)/\mathrm{Pic}^0(C)$
- where $\mathrm{Pic}^0(C)$ acts on $\mathrm{Jac}^d(C_a)$ via the homomorphism $\pi_a^* : \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(C_a)$

Symmetries of the GL_n and PGL_n Hitchin fibration

- for GL_n: fix $a \in \mathcal{A}_{reg}$
- tensor product gives a simply transitive action of $\mathrm{Pic}^0(C_a)$ on $\mathrm{Jac}^d(C_a)$
- $\rightsquigarrow \mathcal{M}_a$ is a torsor for $P_a := \mathrm{Pic}^0(C_a)$
- for PGL_n: fix $a \in \mathcal{A}^0_{reg}$

$$\hat{\mathcal{M}}_a = \mathcal{M}_a / \mathrm{Pic}^0(C)$$

is a torsor for the quotient $\hat{P}_a := P_a / \mathrm{Pic}^0(C)$ abelian variety

Symmetries of the SL_n Hitchin fibration

- recall the spectral cover map $\pi : C_a \to C$
- for an abelian variety A the dual $\hat{A} := \mathrm{Pic}^0(A)$

Definition

For $a \in \mathcal{A}^0_{reg}$ the *norm map* $Nm_{C_a/C} : \mathrm{Pic}^0(C_a) \to \mathrm{Pic}^0(C)$ is defined in any of the following three equivalent ways:

1. D divisor on C_a, $Nm_{C_a/C}(\mathcal{O}(D)) = \mathcal{O}(\pi_* D)$
2. For $L \in \mathrm{Pic}^0(C_a)$ define $Nm_{C_a/C}(L) = det(\pi_*(L)) \otimes det^{-1}(\pi_* \mathcal{O}_{C_a})$.
3. the norm map is the dual of the pull-back map $\pi_a^* : \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(C_a)$, that is $Nm_{C_a/C} = \check{\pi} : \mathrm{Pic}^0(C_a) \cong \check{\mathrm{Pic}}^0(C_a) \to \check{\mathrm{Pic}}^0(C) \simeq \mathrm{Pic}^0(C)$.

- the Prym variety $\mathrm{Prym}^0(C_a) := \ker(Nm_{C_a/C})$ acts on $\mathrm{Prym}^d(C_a) = \check{\mathcal{M}}_a \rightsquigarrow \check{\mathcal{M}}_a$ is a torsor for $\check{P}_a := \mathrm{Prym}^0(C_a)$.
- for PGL_n: $\hat{\mathcal{M}}_a$ is a torsor for $\hat{P}_a = \mathrm{Pic}^0(C_a)/\mathrm{Pic}^0(C) \cong \mathrm{Prym}^0(C_a)/\Gamma \cong \check{P}_a/\Gamma$

Duality of the Hitchin fibres

- short exact sequence of abelian varieties:

$$0 \to \mathrm{Prym}^0(C_a) \hookrightarrow \mathrm{Pic}^0(C_a) \overset{Nm_{C_a/C}}{\twoheadrightarrow} \mathrm{Pic}(C) \to 0$$

- the dual sequence is

$$0 \leftarrow \check{\mathrm{Prym}}^0(C_a) \twoheadleftarrow \mathrm{Pic}^0(C_a) \overset{\pi^*}{\hookleftarrow} \mathrm{Pic}(C) \leftarrow 0 \ ,$$

- $\rightsquigarrow \check{\check{P}}_a = \mathrm{Pic}^0(C_a)/\mathrm{Pic}(C) = \hat{P}_a, \Rightarrow \check{P}_a$ and $\hat{P}_a$ are dual abelian varieties

Theorem (Hausel-Thaddeus, 2003)

For a regular $a \in \mathcal{A}^0_{reg}$ $\check{\mathcal{M}}_a$ and $\hat{\mathcal{M}}_a$ are torsors for dual Abelian varieties (namely $\check{P}_a$ and $\hat{P}_a$).

Strominger-Yau-Zaslow for $\check{\mathcal{M}}_{\mathrm{DR}}$ and $\hat{\mathcal{M}}_{\mathrm{DR}}$

- $$\begin{array}{ccccc} \check{\mathcal{M}}^d & & & & \hat{\mathcal{M}}^e \\ & \overset{\check{\chi}}{\searrow} & & \overset{\hat{\chi}}{\swarrow} & \\ & & \mathcal{A}^0 & & \end{array}$$
- generic fibers are torsors for dual Abelian varieties
- as $\check{\chi}$ and $\hat{\chi}$ are integrable systems $\Rightarrow$ the fibers are complex Lagrangian (i.e. $\omega^c = \omega_J + i\omega_K$ is zero on the fibers)
- [Hitchin, 1987] shows that $\check{\mathcal{M}}$ is hyperkähler and $(\check{\mathcal{M}}, J)$ is the moduli space $\check{\mathcal{M}}_{\mathrm{DR}}$ of (twisted) flat SL_n-connections on C
- $\rightsquigarrow$ $$\begin{array}{ccccc} \check{\mathcal{M}}^d_{\mathrm{DR}} & & & & \hat{\mathcal{M}}^e_{\mathrm{DR}} \\ & \overset{\check{\chi}}{\searrow} & & \overset{\hat{\chi}}{\swarrow} & \\ & & \mathcal{A}^0 & & \end{array}$$
- the fibers of $\check{\chi}$ on $\check{\mathcal{M}}_{\mathrm{DR}}$ now are special Lagrangian because both ω_J and $\mathrm{Im}((\omega_K + i\omega_I)^{2d})$ restrict to zero on the fibers
- Strominger-Yau-Zaslow is satisfied for $\check{\mathcal{M}}_{\mathrm{DR}}$ and $\hat{\mathcal{M}}_{\mathrm{DR}}$!

E-polynomials

- (Deligne 1972) constructs weight filtration $W_0 \subset \cdots \subset W_k \subset \cdots \subset W_{2d} = H_c^d(X;\mathbb{Q})$ for any complex algebraic variety X, plus a pure Hodge structure on W_k/W_{k-1} of weight k
- we say that the weight filtration is *pure* when $W_k/W_{k-1}(H_c^i(X)) \neq 0 \Rightarrow k = i$; examples include smooth projective varieties, $\mathring{\hat{\mathcal{M}}}^d$ and $\mathring{\hat{\mathcal{M}}}^d_{\mathrm{DR}}$
- define $E(X;x,y) := \sum_{i,j,d} (-1)^d x^i y^j h^{i,j}\left(W_k/W_{k-1}(H_c^d(X,\mathbb{C}))\right)$
- basic properties:
  additive - if $X_i \subset X$ locally closed s.t. $\dot{\cup} X_i = X$ then $E(X;x,y) = \sum E(X_i;x,y)$
  multiplicative - $F \to E \to B$ locally trivial in the Zariski topology $E(E;x,y) = E(B;x,y)E(F;x,y)$
- when weight filtration is pure then $E(X;-x,-y) = \sum_{p,q} h^{p,q}(H_c^{p+q}(X))x^p y^q$ is the Hodge $E(X;t,t)$ is the Poincaré polynomial

Stringy E-polynomials

- let finite group Γ act on a non-singular complex variety M
- $E_{st}(M/\Gamma; x, y) := \sum_{[\gamma]\in[\Gamma]} E(M_\gamma/C(\gamma); x, y)(xy)^{F(\gamma)}$
  stringy E-polynomial
- $F(\gamma)$ is the fermionic shift, defined as $F(\gamma) = \sum w_i$, where γ acts on $TX|_{X_\gamma}$ with eigenvalues $e^{2\pi i w_i}$, $w_i \in [0, 1)$
- $F(\gamma)$ is an integer when M is CY and Γ acts trivially on K_M
- motivating property [Kontsevich 1995] if $f : X \to M/\Gamma$ crepant resolution $\Leftrightarrow K_X = f^* K_{M/\Gamma}$ then
  $E(X; x, y) = E_{st}(M/\Gamma; x, y)$
- if B is a Γ-equivariant flat $\mathrm{U}(1)$-gerbe on M, then on each M_γ we get an automorphism of $B|_{M_\gamma} \rightsquigarrow C(\gamma)$-equivariant local system $L_{B,\gamma}$
- we can define
  $E^B_{st}(M/\Gamma; x, y) := \sum_{[\gamma]\in[\Gamma]} E(M_\gamma, L_{B,\gamma}; x, y)^{C(\gamma)}(xy)^{F(\gamma)}$
  stringy E-polynomial twisted by a gerbe

Topological mirror symmetry conjecture - unravelled

Conjecture (Hausel–Thaddeus, 2003)

$(d,n)=(e,n)=1$; $\hat{B}$ *the canonical* Γ*-equivariant gerbe on* $\check{\mathcal{M}}^e_{\mathrm{DR}}$

$$E(\check{\mathcal{M}}^d_{\mathrm{DR}}) = E^{\hat{B}^d}_{st}(\hat{\mathcal{M}}^e_{\mathrm{DR}}) \quad \Leftrightarrow \quad E(\check{\mathcal{M}}^d) = E^{\hat{B}^d}_{st}(\hat{\mathcal{M}}^e)$$

- Theorem for $n=2,3$ using [Hitchin 1987] and [Gothen 1994].
- as Γ acts on $H^*(\check{\mathcal{M}}^d)$ we have $\rightsquigarrow$ $H^*(\check{\mathcal{M}}^d) \cong \bigoplus_{\kappa\in\hat{\Gamma}} H^*_\kappa(\check{\mathcal{M}}^d) \rightsquigarrow$

$$E(\check{\mathcal{M}}^d) = \sum_{\kappa\in\hat{\Gamma}} E_\kappa(\check{\mathcal{M}}^d) = E_0(\check{\mathcal{M}}^d) + \overbrace{\sum_{\kappa\in\hat{\Gamma}^*} E_\kappa(\check{\mathcal{M}}^d)}^{variant}$$

$$\|$$

$$E^{B^d}_{st}(\hat{\mathcal{M}}^e) = \sum_{\gamma\in\Gamma} E(\check{\mathcal{M}}^e_\gamma, L_{B^d,\gamma})^\Gamma = E(\check{\mathcal{M}}^e)^\Gamma + \underbrace{\sum_{\gamma\in\Gamma^*} E(\check{\mathcal{M}}^e_\gamma/\Gamma, L_{B^d,\gamma})}_{stringy}$$

- $\Gamma \cong H^1(C,\mathbb{Z}_n)$ and wedge product induces $w:\Gamma\cong\hat{\Gamma}$
- refined Topological Mirror Test for $w(\gamma)=\kappa$: $E_\kappa(\check{\mathcal{M}}^d) = E(\check{\mathcal{M}}^d_\gamma/\Gamma, L_{B,\gamma})q^{F(\gamma)}$

Example SL_2

- fix $n = 2$ $d = 1$
- $\mathbb{T} := \mathbb{C}^\times$ acts on $\check{\mathcal{M}}$ by $\lambda \cdot (E, \phi) \mapsto (E, \lambda \cdot \phi)) \overset{Morse}{\rightsquigarrow}$

$$H^*(\check{\mathcal{M}}) = \bigoplus_{F_i \subset \check{\mathcal{M}}^{\mathbb{T}}} H^{*+\mu_i}(F_i) \qquad \text{as } \Gamma\text{-modules}$$

- $F_0 = \check{\mathcal{N}}$ where $\phi = 0$; then [Harder–Narasimhan 1975] $\Rightarrow$ $H^*(F_0)$ is trivial Γ-module
- for $i = 1, \ldots, g - 1$

$$F_i = \{(E, \phi) \mid E \cong L_1 \oplus L_2, \phi = \begin{pmatrix} 0 & 0 \\ \varphi & 0 \end{pmatrix}, \varphi \in H^0(L_1^{-1} L_2 K)\}$$

$\rightsquigarrow F_i \to S^{2g-2i-1}(C)$ Galois cover with Galois group Γ

Theorem (Hitchin 1987)

The Γ action on $H^*(F_i)$ is only non-trivial in the middle degree $2g - 2i - 1$. For $\kappa \in \hat{\Gamma}^*$ we have

$$\dim H_\kappa^{2g-2i-1}(F_i) = \binom{2g-2}{2g-2i-1}.$$

Example PGL_2

- $\gamma \in \Gamma = \mathrm{Pic}^0(C)[2] \rightsquigarrow C_\gamma \xrightarrow{2:1} C$ with Galois group $\mathbb{Z}_2$
- $$\begin{array}{ccccc} \mathcal{M}(\mathrm{GL}_1, C_\gamma) \cong T^* \mathrm{Jac}^d(C_\gamma) & \xrightarrow{\text{push-forward}} & \mathcal{M}^d & \supset & \check{\mathcal{M}}^d \\ \| & & \downarrow \det & & \\ T^* \mathrm{Jac}^d(C_\gamma) & \xrightarrow{Nm_{C_\gamma/C}} & T^* \mathrm{Jac}^d(C) & \ni & (\Lambda, 0) \end{array}$$
- let $\check{\mathcal{M}}(GL_1, C_\gamma) := Nm^{-1}_{C_\gamma/C}(\Lambda, 0)$ *endoscopic* H_γ-Higgs moduli space
- after [Narasimhan–Ramanan, 1975] $\check{\mathcal{M}}_\gamma = \check{\mathcal{M}}(GL_1, C_\gamma)/\mathbb{Z}_2 \cong T^* \mathrm{Prym}^d(C_\gamma/C)$
- can calculate $\dim H^{2g-2i+1}(\check{\mathcal{M}}_\gamma/\Gamma, L_{\hat{B},\gamma}) = \binom{2g-2}{2g-2i-1}$ and 0 otherwise

Theorem (Hausel–Thaddeus, 2003)

when $n = 2$ *and* $\kappa = w(\gamma)$
$E_\kappa(\check{\mathcal{M}}; u, v) = E(\check{\mathcal{M}}_\gamma/\Gamma; L_{B,\gamma}, u, v)(uv)^{F(\gamma)}$

- the GL_n-character variety:

$$\mathcal{M}_B^d := \{(A_i, B_i)_{i=1..g} \in \mathrm{GL}_n^{2g} \mid [A_1, B_1] \dots [A_g, B_g] = \zeta_n^d I_n\} // PGL_n$$

non-singular, affine

- the SL_n-character variety:

$$\check{\mathcal{M}}_B^d := \{(A_i, B_i)_{i=1..g} \in \mathrm{SL}_n^{2g} \mid [A_1, B_1] \dots [A_g, B_g] = \zeta_n^d I_n\} // \mathrm{PGL}_n$$

non-singular, affine

- for PGL_n note that $(\mathbb{C}^\times)^{2g}$ acts on $\mathcal{M}_B^d$ and
$\Gamma \cong (\mathbb{Z}_n)^{2g} \subset (\mathbb{C}^\times)^{2g}$ acts on $\check{\mathcal{M}}^d$
$\hat{\mathcal{M}}_B^d := \check{\mathcal{M}}_B^d / \Gamma \cong \mathcal{M}_B^d / (\mathbb{C}^\times)^{2g}$ is an affine orbifold

Theorem (Non-Abelian Hodge Theorem; Simpson, Corlette)

$$\hat{\check{\mathcal{M}}}^d_{\mathrm{Dol}} \overset{diff}{\cong} \hat{\check{\mathcal{M}}}_{\mathrm{DR}} \overset{RH}{\cong} \hat{\check{\mathcal{M}}}_{\mathrm{B}}$$

- RH is complex analytic $\cong$; so SYZ satisfied by $\check{\mathcal{M}}^d_{\mathrm{B}}$ and $\hat{\mathcal{M}}^d_{\mathrm{B}}$

Conjecture (Hausel-Villegas, 2004)

$$(d, n) = (e, n) = 1 \qquad E(\check{\mathcal{M}}_B^d; u, v) = E_{st}^{\hat{B}^d}(\hat{\mathcal{M}}_B^e; u, v)$$

Arithmetic technique to calculate E-polynomials

- E-polynomial of a complex variety X:
  $E(X; u, v) = \sum_{i,p,q}(-1)^i h^{p,q}(Gr_k^W H_c^i(X))u^p v^q$
  where $W_0 \subseteq W_1 \subseteq \ldots \subseteq W_i \subseteq \ldots \subseteq W_{2k} = H_c^k(X)$ is the weight filtration.
- $\mathring{\mathcal{M}}_B$ have a Hodge-Tate type MHS i.e. $h^{p,q} \neq 0$ unless $p = q$
  $E(X; u, v) = E(X, uv) := \sum_{i,k}(-1)^i \dim(Gr_k^W H_c^i(X))(uv)^k$,
  but the MHS is not pure, i.e $k \neq i$ when $h^{k/2,k/2} \neq 0$.
- $X/\mathbb{Z}$ has *polynomial-count*, if
  $E(q) = |X(\mathbb{F}_q)| \in \mathbb{Q}[q]$ is polynomial in q.

Theorem (Katz, 2006)

When $X/\mathbb{Z}$ has polynomial-count $E(X/\mathbb{C}, q) = |X(\mathbb{F}_q)|$

- $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ over $\mathbb{Z}$ as the subscheme $\{xy = 1\}$ of $\mathbb{A}^2$. Then
  $E(\mathbb{C}^*; q) = |(\mathbb{F}_q^*)| = q - 1$
- since $H_c^2(\mathbb{C}^*)$ has weight q and $H_c^1(\mathbb{C}^*)$ has weight 1 $\rightsquigarrow$ checks with Katz

Arithmetic harmonic analysis on $\hat{\check{\mathcal{M}}}_{\mathrm{B}}$

- for any finite group G, [Frobenius 1896], ..., ..., TQFT [Freed–Quinn 1993] $\rightsquigarrow$

$$\left|\left\{a_1, b_1, \dots, a_g, b_g \in G \middle| \prod [a_i, b_i] = z\right\}\right| = \sum_{\chi \in Irr(G)} \frac{|G|^{2g-1}}{\chi(1)^{2g-1}} \chi(z)$$

- when $\zeta_n \in \mathbb{F}_q^*$, i.e $n|q-1$, we get

$$E(\mathcal{M}_{\mathrm{B}}; q) \overset{Katz}{=} |\mathcal{M}_{\mathrm{B}}^d(\mathbb{F}_q)| = (q-1) \sum_{\chi \in Irr(\mathrm{GL}_n(\mathbb{F}_q))} \frac{|\mathrm{GL}_n(\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-2}} \cdot \frac{\chi(\zeta_n^d \cdot I)}{\chi(1)}$$

$Irr(\mathrm{GL}_n(\mathbb{F}_q))$ described combinatorially by [Green, 1955] $\rightsquigarrow$ formula for $E(\mathcal{M}_{\mathrm{B}}; q)$ [Hausel–Villegas, 2008]

- when $n|q-1$

$$E(\check{\mathcal{M}}_{\mathrm{B}}; q) \overset{Katz}{=} |\check{\mathcal{M}}_{\mathrm{B}}^d(\mathbb{F}_q)| = \sum_{\chi \in Irr(\mathrm{SL}_n(\mathbb{F}_q))} \frac{|\mathrm{SL}_n(\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-2}} \cdot \frac{\chi(\zeta_n^d \cdot I)}{\chi(1)}$$

$Irr(\mathrm{SL}_n(\mathbb{F}_q))$ more difficult; only need value of $\chi(\zeta_n^d \cdot I) \rightsquigarrow$ Clifford theory $\rightsquigarrow$ calculation of $E(\check{\mathcal{M}}_{\mathrm{B}}; q)$ by [Mereb, 2010]

Character table of $\mathrm{GL}_2(\mathbb{F}_q)$

Table 1: characters of $\mathbf{GL}_2(\mathbb{F}_q)$
(note that $|\mathbf{GL}_2(\mathbb{F}_q)| = q(q-1)^2(q+1)$)

Classes	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ $a \in \mathbb{F}_q^\times$	$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ $a, b \in \mathbb{F}_q^\times$ $a \neq b$	$\begin{pmatrix} x & 0 \\ 0 & {}^F x \end{pmatrix}$ $x \in \mathbb{F}_{q^2}^\times$ $x \neq {}^F x$	$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ $a \in \mathbb{F}_q^\times$
Number of classes of this type	$q-1$	$\frac{(q-1)(q-2)}{2}$	$\frac{q(q-1)}{2}$	$q-1$
Cardinal of the class	1	$q(q+1)$	$q(q-1)$	q^2-1
$R_{\mathbf{T}}^{\mathbf{G}}(\alpha, \beta)$ $\alpha, \beta \in \mathrm{Irr}(\mathbb{F}_q^\times)$ $\alpha \neq \beta$	$(q+1)\alpha(a)\beta(a)$	$\alpha(a)\beta(b)+$ $\alpha(b)\beta(a)$	0	$\alpha(a)\beta(a)$
$-R_{\mathbf{T}_s}^{\mathbf{G}}(\omega)$ $\omega \in \mathrm{Irr}(\mathbb{F}_{q^2}^\times)$ $\omega \neq \omega^q$	$(q-1)\omega(a)$	0	$-\omega(x) - \omega({}^F x)$	$-\omega(a)$
$\mathrm{Id}_{\mathbf{G}}.(\alpha \circ \det)$ $\alpha \in \mathrm{Irr}(\mathbb{F}_q^\times)$	$\alpha(a^2)$	$\alpha(ab)$	$\alpha(x.{}^F x)$	$\alpha(a^2)$
$\mathrm{St}_{\mathbf{G}}.(\alpha \circ \det)$ $\alpha \in \mathrm{Irr}(\mathbb{F}_q^\times)$	$q\alpha(a^2)$	$\alpha(ab)$	$-\alpha(x.{}^F x)$	0

Character table of $\mathrm{SL}_2(\mathbb{F}_q)$

Table 2: characters of $\mathbf{SL}_2(\mathbb{F}_q)$ for q odd
(note that $|\mathbf{SL}_2(\mathbb{F}_q)| = q(q-1)(q+1)$)

Classes	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ $a \in \{1,-1\}$	$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ $a \in \mathbb{F}_q^\times$ $a \neq \{1,-1\}$	$\begin{pmatrix} x & 0 \\ 0 & {}^F x \end{pmatrix}$ $x.{}^F x = 1$ $x \neq {}^F x$	$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ $a \in \{1,-1\}$, $b \in \{1,x\}$ with $x \in \mathbb{F}_q^\times - (\mathbb{F}_q^\times)^2$
Number of classes of this type	2	$(q-3)/2$	$(q-1)/2$	4
Cardinal of the class	1	$q(q+1)$	$q(q-1)$	$(q^2-1)/2$
$R_{\mathbf{T}}^{\mathbf{G}}(\alpha)$ $\alpha \in \mathrm{Irr}(\mathbb{F}_q^\times)$ $\alpha^2 \neq \mathrm{Id}$	$(q+1)\alpha(a)$	$\alpha(a) + \alpha(\frac{1}{a})$	0	$\alpha(a)$
$\chi_{\alpha_0}^{\varepsilon}$ $\varepsilon \in \{1,-1\}$	$\frac{q+1}{2}\alpha_0(a)$	$\alpha_0(a)$	0	$\frac{\alpha_0(a)}{2}(1- \varepsilon\alpha_0(ab)\sqrt{\alpha_0(-1)q})$
$-R_{\mathbf{T}_s}^{\mathbf{G}}(\omega)$ $\omega \in \mathrm{Irr}(\mu_{q+1})$ $\omega^2 \neq \mathrm{Id}$	$(q-1)\omega(a)$	0	$-\omega(x) - \omega({}^F x)$	$-\omega(a)$
$\chi_{\omega_0}^{\varepsilon}$ $\varepsilon \in \{1,-1\}$	$\frac{q-1}{2}\omega_0(a)$	0	$-\omega_0(x)$	$\frac{\omega_0(a)}{2}(-1+ \varepsilon\alpha_0(ab)\sqrt{\alpha_0(-1)q})$
$\mathrm{Id}_{\mathbf{G}}$	1	1	1	1
$\mathrm{St}_{\mathbf{G}}$	q	1	-1	0

Betti - Topological Mirror Test for $n = 2$

- can calculate $E_{var}(\check{\mathcal{M}}_{\mathrm{B}}) = E(\check{\mathcal{M}}_{\mathrm{B}}) - E(\hat{\mathcal{M}}_{\mathrm{B}}) = E(\check{\mathcal{M}}_{\mathrm{B}}) - E(\mathcal{M}_{\mathrm{B}})/(q-1)^{2g} = (2^{2g}-1)q^{2g-2}\left(\frac{(q-1)^{2g-2}-(q+1)^{2g-2}}{2}\right) = \sum_{i=1}^{g-1}(2^{2g}-1)\binom{2g-2}{2i-1}q^{2g-3+2i}$
- $\check{\mathcal{M}}_{\mathrm{B}}^{\gamma}$ can be identified with $(\mathbb{C}^{\times})^{2g-2}$ and the Γ-equivariant local system $L_{B,\gamma}$ can be explicitly determined $\rightsquigarrow$ $E(\check{\mathcal{M}}_{\mathrm{B}}^{\gamma}/\Gamma, L_{B,\gamma}) = \frac{(q-1)^{2g-2}-(q+1)^{2g-2}}{2}$
- $\Rightarrow E(\check{\mathcal{M}}_{\mathrm{B}}) = E_{st}^{B}(\hat{\mathcal{M}}_{\mathrm{B}})$ when $n = 2$ due to certain patterns in $Irr(\mathrm{SL}_2(\mathbb{F}_q))$ [Schur, 1907] vs. $Irr(\mathrm{GL}_2(\mathbb{F}_q))$ [Jordan, 1907]
- similar argument works when n is a prime;
  for general n one can determine $E(\check{\mathcal{M}}_{\gamma}/\Gamma, L_{B,\gamma})$
  using formulas of Laumon–Ngô and Deligne
- seems to check the Betti-TMS $\rightsquigarrow$
  work in progress with Villegas and Mereb
- $E(\check{\mathcal{M}}_{\mathrm{B}})_{\kappa} = E(\check{\mathcal{M}}_{\mathrm{B}}^{\gamma}, L_{B,\gamma})q^{F(\gamma)} \rightsquigarrow$ twisted κ-character formula for $\mathrm{SL}_n(\mathbb{F}_q)$ = character formula for endoscopy group H_{γ}
  $\rightsquigarrow$ related to Fundamental Lemma?

Hard Lefschetz for Weight and Perverse Filtrations

- $E(\hat{\mathring{\mathcal{M}}}_{\mathrm{B}};1/q) = q^d E(\hat{\mathring{\mathcal{M}}}_{\mathrm{B}};q) \Leftarrow$ Alvis-Curtis in $Irr(G(\mathbb{F}_q))$
- Weight filtration: $W_0 \subset \cdots \subset W_i \subset \cdots \subset W_{2k} = H^k(X)$
- palindromicity $\rightsquigarrow$ Curious Hard Lefschetz Conjecture:

$$L^l : \begin{array}{ccc} Gr^W_{d-2l}(H^{i-l}(\mathcal{M}_{\mathrm{B}})) & \stackrel{\cong}{\to} & Gr^W_{d+2l}H^{i+l}(\mathcal{M}_{\mathrm{B}}) \\ x & \mapsto & x \cup \alpha^l \end{array},$$

  where $\alpha \in W_4H^2(\mathcal{M}_{\mathrm{B}})$
- Perverse filtration: $P_0 \subset \cdots \subset P_i \subset \dots P_k(X) \cong H^k(X)$
  for $f : X \to Y$ proper X smooth Y affine
  (de Cataldo-Migliorini, 2008):
  take $Y_0 \subset \cdots \subset Y_i \subset \dots Y_d = Y$
  s.t. Y_i generic with $\dim(Y_i) = i$ then

$$P_{k-i-1}H^k(X) = \mathrm{ker}(\mathrm{H^k(X) \to H^k(f^{-1}(Y_i)))}$$

- the Relative Hard Lefschetz Theorem holds:

$$L^l : \begin{array}{ccc} Gr^P_{d-l}(H^*(X)) & \stackrel{\cong}{\to} & Gr^P_{d+l}H^{*+2l}(X) \\ x & \mapsto & x \cup \alpha^l \end{array}$$

  where $\alpha \in W_2H^2(X)$ is a relative ample class

$P = W$ conjecture

- recall Hitchin map $\begin{array}{rccc} \chi: & \mathcal{M}_{\mathrm{Dol}} & \to & \mathcal{A} \\ & (E,\phi) & \mapsto & \mathrm{charpol}(\phi) \end{array}$ is proper, thus induces perverse filtration on $H^*(\mathcal{M}_{\mathrm{Dol}})$

Conjecture ("P=W", de Cataldo-Hausel-Migliorini 2008)

$P_k(\mathcal{M}_{\mathrm{Dol}}) \cong W_{2k}(\mathcal{M}_{\mathrm{B}})$ *under the isomorphism* $H^*(\mathcal{M}_{\mathrm{Dol}}) \cong H^*(\mathcal{M}_{\mathrm{B}})$ *from non-Abelian Hodge theory.*

Theorem (de Cataldo-Hausel-Migliorini 2009)

$P = W$ *when* $G = \mathrm{GL}_2, \mathrm{PGL}_2$ *or* SL_2.

- Define $PE(\mathcal{M}_{\mathrm{Dol}}; x, y, q) := \sum q^k E(Gr^P_k(H^*(\mathcal{M}_{\mathrm{Dol}})); x, y)$
- $PE(\mathcal{M}_{\mathrm{Dol}}; \frac{1}{x}, \frac{1}{y}, 1)(xy)^d = E(\mathcal{M}_{\mathrm{Dol}}; x, y) = E(\mathcal{M}_{\mathrm{DR}}; x, y)$
- Conjecture $P = W \Rightarrow PE(\mathcal{M}_{\mathrm{Dol}}; 1, 1, q) = E(\mathcal{M}_{\mathrm{B}}; q)$
- RHL $\rightsquigarrow PE(\mathcal{M}_{\mathrm{Dol}}; x, y, q) = (xyq)^d PE(\mathcal{M}_{\mathrm{Dol}}; x, y; \frac{1}{qxy}) \rightsquigarrow$

Conjecture (Topological Mirror test, TMS)

$$PE^{B^e}_{\mathrm{st}}\left(\check{\mathcal{M}}^d_{\mathrm{Dol}}; x, y, q\right) = (xyq)^d PE^{\hat{B}^d}_{\mathrm{st}}\left(\hat{\mathcal{M}}^e_{\mathrm{Dol}}; x, y, \frac{1}{qxy}\right)$$

Heuristics from S-duality to TMS

- recall that (Kapustin-Witten 2006) suggest that the Geometrical Langlands program is S-duality (reduced from 4 to 2 dimensions)
- $\rightsquigarrow$ it is just mirror symmetry between $\check{\mathcal{M}}_{\mathrm{DR}}$ and $\hat{\mathcal{M}}_{\mathrm{DR}}$
- (Kontsevich 1994)'s homological mirror symmetry proposal $\Rightarrow$ $\mathcal{D}^b(Coh(\check{\mathcal{M}}_{\mathrm{DR}})) \sim \mathcal{D}^b(Fuk(\hat{\mathcal{M}}_{\mathrm{DR}}))" \sim " \mathcal{D}^b(\mathrm{D}mod(Bun_{\mathrm{PGL}_n}))$ (c.f. (Nadler-Zaslow 2006))
- $\rightsquigarrow$ Geometric Langlands program of (Beilinson-Drinfeld 1995)
- $\overset{semi-classical}{\rightsquigarrow} \mathcal{D}^b(Coh(\check{\mathcal{M}}^d_{\mathrm{Dol}})) \sim \mathcal{D}^b(Coh(\hat{\mathcal{M}}^d_{\mathrm{Dol}}))$ $\rightsquigarrow$ fibrewise Fourier-Mukai transform?
- Fibrewise Fourier-Mukai transform should identify $S : H^{r,s}_p(\check{\mathcal{M}}_{\mathrm{Dol}}) \cong H^{r+d/2-p,s+d/2-p}_{st,d-p}(\hat{\mathcal{M}}_{\mathrm{Dol}})$ (Theorem over $\mathcal{A}^0_{\mathrm{reg}}$ (de Cataldo–Hausel–Migliorioni 2010))
- thus S-duality should imply TMS:

$$PE^{B^e}_{\mathrm{st}}\left(\check{\mathcal{M}}^d_{\mathrm{Dol}}; x, y, q\right) = (xyq)^d PE^{\hat{B}^d}_{\mathrm{st}}\left(\hat{\mathcal{M}}^e_{\mathrm{Dol}}; x, y, \frac{1}{qxy}\right)$$

Heuristics from TMS to FL

- let $\underline{\Gamma}$ be the finite group scheme $\pi_0(\check{P}_{\rm ell})$ over $\mathcal{A}^0_{\rm ell}$ let $\kappa \in \hat{\underline{\Gamma}}^*$ (recall $\check{P}_a$ is the Prym variety of $C_a \to C$)
- (Ngô 2008) proves the fundamental lemma in the Langlands program by proving a cohomological statement for the elliptic part of the Hitchin fibration, which in the SL_n-case is a sheaf version of $H^*_p(\tilde{\check{\mathcal{M}}}_{\rm ell})_\kappa \cong H^{*+r}_{p+r}(\mathcal{M}_{\rm ell}(H_\kappa))_{st}$
- there is a natural map $f : \Gamma = Jac_C[n] \cong \mathbb{Z}_n^{2g} \to \underline{\Gamma}$
- $\rightsquigarrow$ the action of Γ on $H^*(\check{\mathcal{M}}_{\rm ell})$ filters through the action of $\underline{\Gamma}$
- $\rightsquigarrow$ $\kappa \in \hat{\underline{\Gamma}}$ defines $\kappa f \in \hat{\Gamma}$, and $H^*_{\kappa f}(\check{\mathcal{M}}_{\rm ell}) \cong H^*_\kappa(\check{\mathcal{M}}_{\rm ell})$
- the refined TMS restricted to $\check{\mathcal{M}}_{\rm ell} \rightsquigarrow$ $H^*_p(\check{\mathcal{M}}_{\rm ell})_{\kappa f} \cong H^{*+F(\gamma)}_{p+r}(\check{\mathcal{M}}_\gamma/\Gamma)$ where $w(\kappa f) = \gamma$
- $\check{\mathcal{M}}_\gamma$ can be identified with H_γ-endoscopy Hitchin systems
- $\rightsquigarrow$ $\mathrm{TMS}|_{\rm ell} \Leftrightarrow$ FL for SL_n
- $\rightsquigarrow$ Ngô's proof of FL $\rightsquigarrow$ $\mathrm{TMS}|_{\rm ell}$, which when n is prime can be extended to TMS for the whole $\check{\mathcal{M}}$
- if we allow the Higgs field to have values in D such that $deg(D) > 2g-2$, then we can deduce TMS for every n from FL from (Chaudouard-Laumon 2009)