

Enumerative invariants of Higgs moduli spaces IV

joint work with András Szenes

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Lecture 1: Cohomology of Higgs moduli via \mathbb{C}^\times action

Lecture 2: Cohomology of character variety via arithmetic & $P = W$

Lecture 3: Cohomology of quiver varieties and Kac's conjectures

Lecture 4: Equivariant Verlinde algebra for Higgs bundles

Recollection on Frobenius algebras

- \mathbb{K} field of characteristic 0
- *Frobenius algebra*: finite dimensional commutative unital \mathbb{K} -algebra F with non-degenerate pairing \langle, \rangle , which is symmetric $\langle a, bc \rangle = \langle ab, c \rangle$
- e.g. $\mathbb{K}R(G)$ for finite group G or $\mathbb{K}[G]^G$ with convolution
- 1+1D TQFT $\Leftrightarrow Z(S^1) = F$ with pairing \langle, \rangle
- (a_i) basis of orthogonal idempotents then

$$Z(\Sigma_g) = \sum_i \langle a_i, 1 \rangle^{1-g} \text{ Verlinde formula}$$

- e.g. when $F = \mathbb{K}[G]^G$ we get $(\chi|G|^{-\frac{1}{2}})_{\chi \in \hat{G}}$ orthogonal idempotents and

$$Z(\Sigma_g) = \sum_{\chi \in \hat{G}} \left(\frac{|G|^2}{\chi(1)^2} \right)^{g-1} = \frac{1}{|G|} |\text{Hom}(\pi_1(\Sigma_g), G)|$$

- 1+1D Chern-Simons theory with finite gauge group of (Freed–Quinn, 1993)
- another example $SU(n)$ Chern-Simons theory
 \leadsto Verlinde algebra discussed next

- C smooth complex projective curve of genus g
- fix rank $n \in \mathbb{Z}_{>0}$, degree $d \in \mathbb{Z}$ and level $k \in \mathbb{Z}_{>0}$
- \mathcal{N}_n^d moduli space of semi-stable rank n fixed degree d vector bundles on C ; projective and smooth when $(d, n) = 1$
- $L \in \text{Pic}(\mathcal{N}_n^d) \cong \mathbb{Z}$ ample generator of Picard group
- Verlinde formula (1988) for $\dim H^0(\mathcal{N}_n^d; L^k) = \chi(\mathcal{N}_n^d, L^k)$
- e.g. for $n = 2$ $d = 1$

$$\dim H^0(\mathcal{N}_2^1, L^k) = \sum_{j=1}^{2k+1} (-1)^{j+1} \left(\frac{k+1}{\sin^2(\frac{j\pi}{2k+2})} \right)^{g-1} =$$

$$\frac{1}{2} \text{Res}_{z=1} \frac{(4k+4)^g}{(z^{k+1} - z^{-(k+1)})((1-1/z)(1-z))^{g-1}} \frac{dz}{z}$$

- proved for
 - $k=1$ by (Beauville–Narasimhan–Ramanan 1988)
 - $n=2$ by (Szenes, Bertram–Szenes 1993)
 - \vdots
 - in all generality by (Teleman–Woodward, 2009)

- Verlinde formula = partition function of a $1 + 1D$ TQFT
- $1 + 1D$ TQFT determined by a Frobenius algebra
i.e. finite dimensional comm. \mathbb{C} -algebra + symmetric pairing
- $R := R(\mathrm{SU}_n) \cong$ character ring of SU_n
- $R \cong R(T_n)^{S_n} \cong (\mathbb{Z}[z_1, \dots, z_n]/(z_1 \cdots z_n - 1))^{S_n}$
irrep $\chi_\lambda \in \mathrm{Irr}(\mathrm{SU}_n) \leftrightarrow s_\lambda \in R$ Schur function
 $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq 0) \in \mathbb{Z}^{n-1}$
- $\mathrm{Ver}_n^k := \mathbb{C} \otimes_{\mathbb{Z}} R / (s_{(k+1)}, s_{(k+2)}, \dots, s_{(k+n-1)})$ has basis $\{s_\lambda\}_{\lambda_1 \leq k}$
- declaring $\langle s_\lambda, s_{\eta^\dagger} \rangle = \delta_{\lambda\eta} \rightsquigarrow$ non-degenerate pairing

Theorem (Goodman-Wenzl 1990; Gepner 1991; Witten 1993)

$(\mathrm{Ver}_n^k, \langle, \rangle)$ is a Frobenius algebra (i.e. $\langle a, bc \rangle = \langle ab, c \rangle$).
 \cong Verlinde algebra, with partition function giving Verlinde formulae

- geometrization: Borel-Weil-Bott theory \rightsquigarrow
 $\chi_{T_n}(\mathcal{F}; L_\lambda) = s_\lambda \in R(T_n)^{S_n}$ on flag variety $\mathcal{F} := \mathrm{SU}_n/T_n$

3:30 pm Thursday, September 18, 2003

Geometry Seminar: Integrals on the moduli of Higgs bundles

by **Tamas Hausel** (UT - Math) in RLM 9.166

First I explain how one can integrate in the circle equivariant cohomology of the non-compact moduli space of Higgs bundles on a Riemann surface. Then I explain why it is presently impossible to calculate these integrals. Then I mention that the physicists Moore-Nekrasov-Shatavili do have formulas, surprisingly related to certain Bethe Ansatz equations, for the equivariant volumes of these spaces, a result which is obtained using Feynman integral methods. Then I show some calculational results with Andras Szenes which makes us conjecture that a miraculous agreement in the contribution to our integrals will give a general formula for this incalculable calculation, and also gives a new, fairly elementary, approach to Witten's intersection numbers on the moduli space of stable bundles. We will calculate integrals on the Higgs moduli space related to equivariant volumes, χ -genuses and the Verlinde formulas, and finish with a review of possible applications of these ideas.

Submitted by **Vittoria Esile**

- $\mathcal{M}_n^d \supset T^*\mathcal{N}_n^d$ moduli ss rank n fixed degree d Higgs bundles
- i.e. Higgs bundles of the form (E, ϕ) where $\text{rank}(E) = n$, $\det(E) = \Lambda$, $\text{deg}(\Lambda) = d$ and $\phi \in \text{End}_0(E) \otimes K$
- $\mathbb{T} := \mathbb{C}^\times$ acts on \mathcal{M}_n^d by scaling Higgs field
- $L \in \text{Pic}(\mathcal{M}_n^d)$ ample generator with \mathbb{T} action trivial on $L^k|_{\mathcal{N}_n^d}$
- \mathbb{T} acts on $H^0(\mathcal{M}_n^d, L^k)$ with weights ≥ 0
- $\text{grdim}(H^0(\mathcal{M}_n^d, L^k)) = \sum_{i=0}^{\infty} \dim(H^0(\mathcal{M}_n^d, L^k)^i) t^i \in \mathbb{Z}[[t]]$
- (Paradan 2011) \rightsquigarrow

$$\chi_{\mathbb{T}}(\mathcal{M}_n^d, L^k) = \sum_{F_i} \int_{F_i} \text{ch}_{\mathbb{T}}(L^k|_{F_i} \otimes \text{Sym}_t N^* F_i) \text{Todd}(TF_i)$$
- $F_i \subset (\mathcal{M}_n^d)^{\mathbb{T}}$ fixed point components
- (Hausel–Szenes, 2003) direct computation $\rightsquigarrow \chi_{\mathbb{T}}(\mathcal{M}_2^1, L^k) =$

$$\sum_{a=1, t, 1/t} \text{Res}_{z=a} \frac{2^{2g-1} \left[k+1 + \frac{zt}{1-tz} + \frac{t/z}{1-t/z} \right]^g}{\left[z^{k+1} \frac{1-t/z}{1-tz} - z^{-(k+1)} \frac{1-tz}{1-t/z} \right] \left[(1-1/z)(1-z)(1-t/z)(1-tz) \right]^{g-1}} \frac{dz}{z},$$
- (Hausel–Szenes, 2003) conjecture for higher n
- recently (Halpern–Leistner 2016) and (Andersen–Gukov–Pei 2016) gave formulas for $\chi_{\mathbb{T}}(\mathcal{M}_G, L^k)$ for general G building on the work of (Teleman–Woodward 2009)

- by physics arguments (Gukov–Pei 2015)
 $\chi_{\mathbb{T}}(\mathcal{M}_n^d, L^k)$ arises from a 1 + 1D TQFT, i.e. a Frobenius algebra, dubbed *equivariant Verlinde algebra*
- we construct this algebra explicitly:
 t -deforming Goodman–Wenzl presentation of Ver_n^k
- step 1: find t -deformation $R_t(\mathrm{SU}_n)$ of $R(\mathrm{SU}_n)$ and of basis s_λ
- step 2: find deformed ideal I_t in the deformation
 $\rightsquigarrow QVer_n^k := \overline{\mathbb{C}(t)} \otimes_{\mathbb{Z}(t)} R_t(\mathrm{SU}_n) / I_t$
- step 3: define pairing \langle, \rangle_t and show it yields symmetric algebra $(QVer_n^k, \langle, \rangle_t)$
- step 4: check partition function giving $\chi_{\mathbb{T}}(\mathcal{M}_n^d, L^k)$ and compare with properties described in (Gukov–Pei, 2015), (Gukov–Pei–Yan–Ye 2016), (Andersen–Gukov–Pei, 2016)

step 1. Definition of $R_t(\mathrm{SU}_n)$ and χ_λ^t

- recall geometrisation of $R(\mathrm{SU}_n)$:
 $\chi_{T_n}(\mathcal{F}; L_\lambda) = \chi_\lambda \in R(T_n)^{S_n}$ on flag variety $\mathcal{F} := \mathrm{SU}_n/T_n$
- consider $T^*\mathcal{F}$ and $L_\lambda := \pi^*L_\lambda \in \mathrm{Pic}_{\mathbb{T}}(T^*\mathcal{F})$
- $\chi_\lambda^t := \chi_{T_n \times \mathbb{T}}(T^*\mathcal{F}, L_\lambda) \in R(T_n \times \widehat{\mathbb{T}})^{S_n} \cong R(\mathrm{SU}_n)[[t]] =: R_t(\mathrm{SU}_n)$
- χ_λ^t computed by (Gupta/Brylinski 1987):
 $\chi_\lambda^t = E_\lambda \in R(T_n)^{S_n}[[t]]$ can be obtained
 $E_\lambda = t_\lambda(t) P_\lambda / \psi_t$
 $P_\lambda = \frac{\sum_{w \in S_n} (-1)^{\sigma(w)} z^{w(\lambda)} w(\Delta_t)}{\Delta_1 t_\lambda(t)} \in R(\mathrm{SU}_n)[t]$ Hall-Littlewood
 $\Delta_t = z^\rho \prod_{\alpha \in \Phi^-} (1 - tz^\alpha)$; $\psi_t = \prod_{\alpha \in \Phi} (1 - tz^\alpha)$
 $t_\lambda(t) = \sum_{w \in St_{S_n}(\lambda)} t^{l(w)}$

Theorem (Gupta 1987)

$$\langle P_\lambda, E_{\eta^\dagger} \rangle = \frac{1}{n!} \mathrm{Res}_{z=0} \frac{E_\lambda \psi_t}{t_\lambda(t)} E_{\eta^\dagger} \psi_1 \frac{dz}{z} = \delta_{\lambda\eta}$$

step 2. Definition of $QVer_n^k$

- for $\alpha \in \Phi$ define

$$b_\alpha := z^{(k+n)\alpha} \prod_{\beta \in \Phi} (1 - tz^\beta)^{-\langle \alpha, \beta \rangle} \in R(T_n)(t)$$

$$\leadsto \text{non-symmetric Bethe-Ansatz equation } b_\alpha = 1$$
- we have $b_{\alpha+\beta} = b_\alpha b_\beta$ and $b_{w\alpha} = w(b_\alpha)$
- thus $I'_t := (1 - b_{\alpha_1}, \dots, 1 - b_{\alpha_{n-1}}) = (1 - b_\alpha)_{\alpha \in \Phi} \triangleleft R(T_n)(t)$
- then $\text{Spec}(\mathbb{F}R(T_n)/I'_t) \subset T_n(\mathbb{F})$ with $\mathbb{F} := \overline{\mathbb{C}(t)}$
i.e. the solutions of the Bethe-Ansatz equations
 $b_{\alpha_1} = 1, \dots, b_{\alpha_{n-1}} = 1$ in $T_n(\mathbb{F})$ are invariant under S_n
- $\text{Spec}_n^k := \text{Spec}(\mathbb{F}R(T_n)/I'_t) \setminus \text{Spec}(\mathbb{F}R(T_n)/(1 - z^\alpha)_{\alpha \in \Phi})$
- for $\lambda = (k+1 = \lambda_1 \geq \dots \geq \lambda_{n-1} \geq 0)$ form

$$B_\lambda = \frac{\sum_{w \in S_n} (-1)^{\sigma(w)} z^{w(\lambda)} (1 - tz^{w(-\theta)})_{w(\Delta_t)}}{\Delta_1} \in R(T_n)^{S_n}[t]$$
 symmetric Bethe-Ansatz polynomial
then $B_\lambda \Delta_1 \in I'_t$
- $\lambda_m := (k+1, 1, \dots, 1, 0, \dots, 0) = k\omega_1 + \omega_m$
 $I_t := (B_{\lambda_1}, \dots, B_{\lambda_{n-1}}) \triangleleft R(T_n)^{S_n}[t]$
- $QVer_n^k := \mathbb{F}R(T_n)^{S_n}[t]/I_t$
- then $\text{Spec}_n^k/S_n \subset \text{Spec}(QVer_n^k) \subset T_n(\mathbb{F})/S_n$

step 3. $(QVer_n^k, \langle, \rangle_t)$ is a Frobenius algebra

- $\langle E_\lambda, E_{\eta^\dagger} \rangle_t := \delta_{\lambda\eta} \tilde{t}_\lambda(t) (1-t)^{n-1}$, $\tilde{t}_\lambda(t) = \sum_{w \in St_{\xi_n^k}(\lambda)} t^{l(w)}$

Theorem (Hausel–Szenes 2016)

$(E_\lambda)_{\lambda_1 \leq k}$ is a basis of $QVer_n^k$, and $\langle a, bc \rangle_t = \langle ab, c \rangle_t$

- proof: assume $l_1(\lambda), l_1(\eta) \leq k$ and denote

$r = \tau_{k\omega_1} \circ w_{(1,2,\dots,n)} : \mathbb{Z}^{n-1} \rightarrow \mathbb{Z}^{n-1}$ then prove:

$$\langle E_\lambda, E_{\eta^\dagger} \rangle_t / (1-t)^{n-1}$$

$$= \frac{1}{n!} \sum_{a \in \text{Spec}_n^k} \frac{E_\lambda(a) E_{\eta^\dagger}(a) \psi_t(a) \psi_1(a)}{\text{Jac}(\log(b_{\alpha_1}), \dots, \log(b_{\alpha_{n-1}}))(a)}$$

$$= \frac{(-1)^{n-1}}{n!} \sum_{a \in \text{Spec}_n^k} \text{Res}_{z=a} \frac{E_\lambda E_{\eta^\dagger} \psi_t \psi_1}{(1-b_{\alpha_1}) \cdots (1-b_{\alpha_{n-1}})} \frac{dz}{z}$$

$$= \frac{(-1)^{n-1}}{n!} \sum_{a \in \text{Spec}_n^k} \text{Res}_{z=a} \frac{E_{r(\lambda)} E_{r(\eta)^\dagger} \psi_t \psi_1}{(1-b_{\alpha_1}) \cdots (1-b_{\alpha_{n-1}})} \frac{dz}{z}$$

observe for some r^m no residue at infinity \Rightarrow

$$= \frac{1}{n!} \text{Res}_{z=0} \frac{E_{r^m(\lambda)} E_{r^m(\eta)^\dagger} \psi_t \psi_1}{(1-b_{\alpha_1}) \cdots (1-b_{\alpha_{n-1}})} \frac{dz}{z} \stackrel{\text{Gupta}}{=} \delta_{r^m(\lambda)r^m(\eta)} t_{r^m(\lambda)}(t) = \delta_{\lambda\eta} \tilde{t}_\lambda(t)$$

- $\Rightarrow |(\lambda)_{\lambda_1 \leq k}| = \binom{n+k-1}{n-1} \leq \dim R_t / I_t \stackrel{\text{Bezout}}{\leq} \binom{n+k-1}{n-1} \blacksquare$

step 4. Partition function and other checks

- \Rightarrow rotation on $QVer_n^k$ corresponds to multiplying by $\tilde{P}_{k\omega_1} = P_{k\omega_1} t_{k\omega_1}(t) / \tilde{t}_{k\omega_1}(t)$, and $\tilde{P}_{k\omega_1}^d = \tilde{P}_{k(\omega_1+\dots+\omega_d)} = \tilde{P}_{k\omega_{1d}}$

- $Tr := QVer_n^k \rightarrow \mathbb{F}$ by

$$Tr(E) := \frac{1}{n!} \sum_{a \in \text{Spec}_n^k} \frac{E(a) \psi_t(a) \psi_1(a) (1-t)^{n-1}}{\text{Jac}(\log(b_{\alpha_1}), \dots, \log(b_{\alpha_{n-1}}))}$$

then $\Rightarrow \langle E_1, E_2 \rangle_t = Tr(E_1 E_2)$

- we have $Z_n^k(C_g^d) =$

$$= \frac{1}{n!} \sum_{a \in \text{Spec}_n^k} \tilde{P}_{k\omega_{1d}}(a) \left(\frac{\text{Jac}(\log(b_{\alpha_1}), \dots, \log(b_{\alpha_{n-1}}))(a)}{\psi_t(a) \psi_1(a) (1-t)^{n-1}} \right)^{g-1}$$

$$= \frac{(-1)^{n-1}}{n!} \sum_{a \in \text{Spec}_n^k} \text{Res}_{z=a} \frac{\tilde{P}_{k\omega_{1d}} \text{Jac}(\log(b_{\alpha_1}), \dots, \log(b_{\alpha_{n-1}}))^g}{(1-b_{\alpha_1}) \cdots (1-b_{\alpha_{n-1}}) (\psi_t \psi_1 (1-t)^{n-1})^{g-1}} \frac{dz}{z}$$

$$= \frac{1}{n!} \sum_{a \in V(\psi_t \psi_1)} \text{Res}_{z=a} \frac{\tilde{P}_{k\omega_{1d}} \text{Jac}(\log(b_{\alpha_1}), \dots, \log(b_{\alpha_{n-1}}))^g}{(1-b_{\alpha_1}) \cdots (1-b_{\alpha_{n-1}}) (\psi_t \psi_1 (1-t)^{n-1})^{g-1}} \frac{dz}{z}$$

- when $n = 2$, $d = 1$ this agrees with (Hausel–Szenes 2003)

$$\sum_{a=1, t, 1/t} \text{Res}_{z=a} \frac{\frac{2^{2g-1}}{(1-t)^{g-1}} \left[k+1 + \frac{t/z}{1-t/z} + \frac{tz}{1-tz} \right]^g}{\left[z^{k+1} \frac{1-t/z}{1-tz} - z^{-(k+1)} \frac{1-tz}{1-t/z} \right] \left[(1-1/z)(1-z)(1-t/z)(1-tz) \right]^{g-1}} \frac{dz}{z}$$

- $QVer_2^k$ and $QVer_3^k$ matches (Gukov–Pei–etal, 2015, 2016)

Geometric information in $\chi_{\mathbb{T}}(\mathcal{M}_n^d, L^k)$

- $\chi: \mathcal{M}_n^d \rightarrow \mathcal{A} \cong \bigoplus_{i=2}^n H^0(C; K_C^i)$ Hitchin map
 $(E, \phi) \mapsto \text{charpol}(\phi)$
- Hitchin map for $\text{GL}_1: (L, \phi) \in \mathcal{M}_{\text{Dol}}^{d;1} \cong J^d(C) \times H^0(C; K_C)$
 χ is projection to second factor:
 $\chi: J^d(C) \times H^0(C; K_C) \rightarrow H^0(C; K_C)$
- in general χ is a completely integrable Hamiltonian system
 generic fibre $\chi^{-1}(a)$ is isomorphic with an Abelian variety
 e.g. $\chi^{-1}(a) \cong \text{Prym}(C_a/C)$ for SL_n
- $V := \chi_*(L^k)$ is a \mathbb{T} -equivariant vector bundle on $\mathcal{A} \rightsquigarrow$
 $V = V_0 \otimes \mathcal{O}_{\mathcal{A}}$ trivial \mathbb{T} -bundle (Masuda–Moser–Jauslin–Petrie 1995)
- $\text{char}(V_0) = \chi_{\mathbb{T}}(\mathcal{C}_n^d; L^k) \in \mathbb{N}[t] \rightsquigarrow \text{char}(V_0)(0) = \chi(\mathcal{N}_n^d, L^k)$
 $V_a = \chi_{\mathbb{T}}(\chi^{-1}(a), L^k)$ for generic $a \in \mathcal{A} \rightsquigarrow$
 $\text{rank}(V) = n^g (nk)^{(n^2-1)(g-1)}$
- $\chi_{\mathbb{T}}(\mathcal{M}_n^d, L^k) = \frac{\text{char}(V_0)}{\prod_{i=2}^n (1-t^i)^{(2i-1)(g-1)}}$
- e.g. $g = 2 \rightsquigarrow \chi_{\mathbb{T}}(\mathcal{M}_2^1; L^2) = \frac{49t^3 + 113t^2 + 75t + 19}{(1-t^2)^3}$

- We can analogously define $(QVer_n^k, \langle, \rangle_t)$ for any compact simply-connected semisimple Lie group G . Is it a Frobenius algebra?
- We understand how to read off $\chi(\mathcal{M}_\mu, L^k)$ for parabolic Higgs bundles \mathcal{M}_μ from $QVer_n^k$. How to include irregular singularities? (Fredrickson-Pei-Yan-Ye 2017)
- Hall–Littlewood polynomials deform to Macdonald polynomials. Is there a corresponding further deformation of $QVer_n^k$? What does it compute?
- What TQFT computes equivariant Hirzebruch y -genera of \mathcal{M}_n^d ?
- can we enhance our $1 + 1D$ TQFT to a $2 + 1D$ TQFT deforming the Jones–Witten TQFT?
- Is there a representation theory of deformations of affine Kac-Moody algebras or Hecke algebras/quantum groups at root of unity behind $QVer_n^k$?
- Can we relate $\chi_{\mathbb{T}}(\mathcal{M}_n^d, L^k)$ along the Hitchin map to the abelianization program of (Atiyah–Hitchin 1987)?