Enumerative invariants of Higgs moduli spaces II

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Plan for lectures

Lecture 1: Cohomology of Higgs moduli via $\mathbb{C}^\times$ action

Lecture 2: Cohomology of character variety via arithmetic

Lecture 3: Cohomology of quiver varieties and Kac's conjectures

Lecture 4: Equivariant Verlinde algebra for Higgs bundles

Recollection on Frobenius algebras

- $\mathbb{K}$ field of characteristic 0
- *Frobenius algebra*: finite dimensional commutative unital $\mathbb{K}$-algebra F with non-degenerate pairing $\langle , \rangle$, which is symmetric $\langle a, bc\rangle = \langle ab, c\rangle$
- e.g. $\mathbb{K}R(\mathrm{G})$ for finite group G or $\mathbb{K}[\mathrm{G}]^{\mathrm{G}}$ with convolution
- 1+1D TQFT $\Leftrightarrow$ $Z(S^1) = F$ with pairing $\langle , \rangle$
- (a_i) basis of orthogonal idempotents then

  $Z(\Sigma_g) = \sum_i \langle a_i, 1\rangle^{1-g}$ *Verlinde formula*

- e.g. when $F = \mathbb{K}[\mathrm{G}]^{\mathrm{G}}$ we get $(\chi|G|^{-\frac{1}{2}})_{\chi\in\hat{\mathrm{G}}}$ orthogonal idempotents and

  $$Z(\Sigma_g) = \sum_{\chi\in\hat{\mathrm{G}}} \left(\frac{|\mathrm{G}|^2}{\chi(1)^2}\right)^{g-1} = \frac{1}{|\mathrm{G}|}|Hom(\pi_1(\Sigma_g), \mathrm{G})|$$

- 1+1D Chern-Simons theory with finite gauge group of (Freed–Quinn,1993)
- another example $SU(n)$ Chern-Simons theory $\rightsquigarrow$ Verlinde algebra discussed in Lecture 4

Diffeomorphic spaces in non-Abelian Hodge theory

- C genus g curve; fix $r > 0$

$$\mathcal{M}_{\mathrm{Dol}}^{d,r} := \left\{ \begin{array}{c} \text{moduli space of semistable rank } r \\ \text{degree } d \text{ Higgs bundles } (E,\phi) \\ \phi \in H^0(C, \mathrm{End}(E) \otimes K) \text{ Higgs field} \end{array} \right\}$$

$$\mathcal{M}_{\mathrm{B}}^{d,r} := \{A_1, B_1, .., A_g, B_g \in \mathrm{GL}_r | \prod_{i=1}^{g} A_i^{-1} B_i^{-1} A_i B_i = e^{\frac{2\pi i d}{r}} Id\} /\!\!/ \mathrm{PGL}_r$$

when $(d,r) = 1$ these are smooth non-compact varieties

- Non-Abelian Hodge Theorem: $\mathcal{M}_{\mathrm{Dol}}^{d,r} \overset{diff}{\cong} \mathcal{M}_{\mathrm{B}}^{d,r}$

[Hitchin, Donaldson, Corlette, Simpson]

- Problem: $P_t(\mathcal{M}_{\mathrm{Dol}}^{d,r}) = P_t(\mathcal{M}_{\mathrm{B}}^{d,r})$?

Easy examples

- when $r = 1 \rightsquigarrow$

  $\phi : H^0(C; \mathrm{End}(E) \otimes K_C) \cong H^0(C; K_C) \rightsquigarrow$

  $\mathcal{M}^{d,1}_{\mathrm{Dol}} \cong T^*J^d(C) \cong J^d(C) \times H^0(C; K_C)$

  $\cong_{diff}$

  $\mathcal{M}^{d,1}_{\mathrm{B}} \cong \mathrm{Hom}(\pi_1(C), \mathrm{GL}_1(\mathbb{C})) \cong (\mathbb{C}^\times)^{2g}$
- $r > 1$ and $g = 0$ both $\mathcal{M}^{d,r}_{\mathrm{B}}$ and $\mathcal{M}^{d,r}_{\mathrm{Dol}}$ are empty
- $r > 1$ and $g = 1 \rightsquigarrow$ Stone-von Neumann $\rightsquigarrow$

  $\mathcal{M}^{d,r}_{\mathrm{B}} \cong (\mathbb{C}^*)^2$

  $\cong_{diff}$

  $\mathcal{M}^{d,r}_{\mathrm{Dol}} \cong T^*\mathrm{J}^{\mathrm{d}}(C) \cong T^*C \cong C \times H^0(C; K_C) \cong C \times \mathbb{C}$

Mixed Hodge polynomials

- (Deligne 1971) proved the existence of $W_0 \subset \cdots \subset W_i \subset \cdots \subset W_{2k} = H^k(X;\mathbb{Q})$ for any complex algebraic variety X which is
  - functorial
  - compatible with cup-product
- $H(X;q,t) = \sum \dim(W_i/W_{i-1}(H^k(X)))t^k q^{\frac{i}{2}}$, *mixed Hodge polynomial*
- $P(X;t) = H(X;1,t)$, *Poincaré polynomial*
- $E(X;q) = q^D H(X;1/q,-1)$, *E-polynomial* of X.

Theorem (Katz 2008)

If M is a smooth quasi-projective variety defined over $\mathbb{Z}$ and

$$\#\{M(\mathbb{F}_q)\} = E(q)$$

is a polynomial in q, then $E(M;q) = E(q)$.

Examples for mixed Hodge polynomials

- M smooth semi-projective variety (e.g. $\mathcal{M}_{\mathrm{Dol}}^{d,r}$) $\rightsquigarrow$
  $W_{k-1}(H^k(M)) = 0$ because it is smooth
  $W_k(H^k(M)) = W_k(H^k(C)) = H^k(C) = H^k(M) \Leftarrow C$ projective
  such cohomology is called *pure*

  $\rightsquigarrow H(M; q, t) = P(M; tq^{\frac{1}{2}})$

  $\rightsquigarrow E(M; q) = q^D P(M; -q^{-\frac{1}{2}})$
- $H(T^*J^d(C); q, t) = H(J^d(C); q, t) = (1 + q^{\frac{1}{2}} t)^{2g}$ MHS is pure

  $E(T^*J^d(C); q) = q^g(1 - q^{\frac{1}{2}})^{2g}$ *not* polynomial count
- $\#(\mathrm{GL}_1(\mathbb{F}_q)) = q - 1 \overset{Katz}{\Rightarrow} E(\mathrm{GL}_1(\mathbb{C}); q) = E(\mathbb{C}^\times; q) = q - 1 \rightsquigarrow$
  $W_1 H^1(\mathbb{C}^\times) = 0 \rightsquigarrow H(\mathbb{C}^\times; q, t) = 1 + qt$ MHS *not* pure
- $\rightsquigarrow H((\mathbb{C}^\times)^{2g}; q, t) = (1 + qt)^{2g}$ and $E((\mathbb{C}^\times)^{2g}; q) = (q - 1)^{2g}$
- $E((\mathbb{C}^\times)^{2g}; q)$ palindromic $\Leftrightarrow E((\mathbb{C}^\times)^{2g}; q) = q^{2g} E((\mathbb{C}^\times)^{2g}; 1/q)$
- extends to $H(\mathbb{C}^\times)^{2g}; q, t) = (qt)^{2g} H(\mathbb{C}^\times)^{2g}; 1/qt^2, t)$

Theorem (Frobenius 1896, Hurwitz 1902, Freed-Quinn 1993,...)

Let $z \in \mathrm{G}$ in a finite group G then

$$\frac{1}{|\mathrm{G}|}\#\{a_j, b_j \in \mathrm{G} | [a_1, b_1]\dots[a_g, b_g] = z\} =$$
$$= \sum_{\chi \in \mathrm{Irr}(\mathrm{G})} \frac{|\mathrm{G}|^{2g-2}}{\chi(1)^{2g-2}} \frac{\chi(z)}{\chi(1)}$$

- (Hausel-Villegas 2008) calculates $E(\mathcal{M}_{\mathrm{B}}^{d,r}; q) \overset{Katz}{=} |\mathcal{M}_{\mathrm{B}}^{d,r}(\mathbb{F}_q)| = \sum_{\chi \in Irr(\mathrm{GL}_r(\mathbb{F}_q))} \frac{|\mathrm{GL}_r(\mathbb{F}_q)|^{2g-2}(q-1)}{\chi(1)^{2g-2}} \frac{\chi(\xi_r^d)}{\chi(1)}$
- $\rightsquigarrow E(\mathcal{M}_{\mathrm{B}}^{d,r}; q) = E(\mathcal{M}_{\mathrm{B}}^{d',r}; q)$ when $(d, r) = (d', r) = 1$
- $\mathcal{M}_{\mathrm{B}}^{d,r}$ and $\mathcal{M}_{\mathrm{B}}^{d',r}$ Galois conjugate $\Rightarrow$ $H(\mathcal{M}_{\mathrm{B}}^{d,r}; q, t) = H(\mathcal{M}_{\mathrm{B}}^{d',r}; q, t)$
- in particular $P_t(\mathcal{M}_{\mathrm{B}}^{d,r}) = P_t(\mathcal{M}_{\mathrm{Dol}}^{d,r})$ does not depend on d

Character table of $\mathrm{GL}_2(\mathbb{F}_q)$

Table 1: characters of $\mathbf{GL}_2(\mathbb{F}_q)$
(note that $|\mathbf{GL}_2(\mathbb{F}_q)| = q(q-1)^2(q+1)$)

Classes	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ $a \in \mathbb{F}_q^\times$	$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ $a, b \in \mathbb{F}_q^\times$ $a \neq b$	$\begin{pmatrix} x & 0 \\ 0 & {}^F x \end{pmatrix}$ $x \in \mathbb{F}_{q^2}^\times$ $x \neq {}^F x$	$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ $a \in \mathbb{F}_q^\times$
Number of classes of this type	$q-1$	$\frac{(q-1)(q-2)}{2}$	$\frac{q(q-1)}{2}$	$q-1$
Cardinal of the class	1	$q(q+1)$	$q(q-1)$	q^2-1
$R_{\mathbf{T}}^{\mathbf{G}}(\alpha, \beta)$ $\alpha, \beta \in \mathrm{Irr}(\mathbb{F}_q^\times)$ $\alpha \neq \beta$	$(q+1)\alpha(a)\beta(a)$	$\alpha(a)\beta(b)+$ $\alpha(b)\beta(a)$	0	$\alpha(a)\beta(a)$
$-R_{\mathbf{T}_s}^{\mathbf{G}}(\omega)$ $\omega \in \mathrm{Irr}(\mathbb{F}_{q^2}^\times)$ $\omega \neq \omega^q$	$(q-1)\omega(a)$	0	$-\omega(x) - \omega({}^F x)$	$-\omega(a)$
$\mathrm{Id}_{\mathbf{G}}.(\alpha \circ \det)$ $\alpha \in \mathrm{Irr}(\mathbb{F}_q^\times)$	$\alpha(a^2)$	$\alpha(ab)$	$\alpha(x.{}^F x)$	$\alpha(a^2)$
$\mathrm{St}_{\mathbf{G}}.(\alpha \circ \det)$ $\alpha \in \mathrm{Irr}(\mathbb{F}_q^\times)$	$q\alpha(a^2)$	$\alpha(ab)$	$-\alpha(x.{}^F x)$	0

Example $\mathrm{GL}_2(\mathbb{F}_q)$

- $$\frac{1}{|\mathrm{PGL}_2(\mathbb{F}_q)|(q-1)^{2g}}\#\{a_j, b_j \in \mathrm{GL}_2(\mathbb{F}_q) | [a_1, b_1]\dots[a_g, b_g] = -1\} =$$

$$\frac{1}{|\mathrm{PGL}_2(\mathbb{F}_q)|(q-1)^{2g}} \sum_{\chi \in \mathrm{Irr}(\mathrm{GL}_2(\mathbb{F}_q))} \frac{|\mathrm{GL}_2(\mathbb{F}_q)|^{2g-1}}{\chi(1)^{2g-2}} \frac{\chi(-1)}{\chi(1)} =$$

$$(q^2-1)^{2g-2} + q^{2g-2}(q^2-1)^{2g-2} - \frac{1}{2}q^{2g-2}(q-1)^{2g-2} - \frac{1}{2}q^{2g-2}(q+1)^{2g-2}$$

$$\overset{Katz}{=} \frac{E(\mathcal{M}_{\mathrm{B}}^{d,2})}{(q-1)^{2g}}.$$

- e.g. $g = 0$ gives 0 when $g = 1$ it gives 1

Mixed Hodge polynomials of character varieties

Conjecture (Hausel-Villegas, 2008)

$$\sum_\lambda \prod \frac{\left(z^{2l+1}-w^{2a+1}\right)^{2g}}{(z^{2l+2}-w^{2a})(z^{2l}-w^{2a+2})} T^{|\lambda|} = \exp\left(\sum_{r,k} \frac{H\left(\mathcal{M}_{\mathrm{B}}^{d,r}; w^{2k}, -(zw)^{-2k}\right)(zw)^{D_r}}{(z^{2k}-1)(1-w^{2k})} \frac{T^{rk}}{k}\right)$$

- when $g=0$ $\mathcal{M}_{\mathrm{B}}^{d,r}$ empty unless $r=1$ when $\mathcal{M}_{\mathrm{B}}^{d,1}=pt \overset{HV}{\rightsquigarrow}$

  $$\sum_\lambda \prod \frac{T^{|\lambda|}}{(z^{2l+2}-w^{2a})(z^{2l}-w^{2a+2})} = \exp\left(\sum_{k\geq 1} \frac{1}{(z^{2k}-1)(1-w^{2k})} \frac{T^k}{k}\right)$$

  proved by [Garsia–Haiman, 1996]

- when $g=1$ $\mathcal{M}_{\mathrm{B}}^{d,r}=(\mathbb{C}^*)^2$ by Stone-von Neumann $\overset{HV}{\rightsquigarrow}$

  $$\sum_\lambda \prod \frac{\left(z^{2l+1}-w^{2a+1}\right)^{2}}{(z^{2l+2}-w^{2a})(z^{2l}-w^{2a+2})} T^{|\lambda|} = \exp\left(\sum_{k\geq 1} \frac{(z^k-w^k)^2}{(z^{2k}-1)(1-w^{2k})(1-T^k)} \frac{T^k}{k}\right)$$

  proved geometrically by [Waelder, 2008] and combinatorially by [Carlsson, Villegas 2016] and [Rains, Warnaar 2016]

Formula for $H(\mathcal{M}_{\mathrm{B}}^{d,2}; q, t)$

- when $r = 2$ Conjecture $\rightsquigarrow$

$$\frac{H_2(q,t)}{(1+qt)^{2g}} = \frac{(q^2t^3+1)^{2g}}{(q^2t^2-1)(q^2t^4-1)} + \frac{q^{2g-2}t^{4g-4}(q^2t+1)^{2g}}{(q^2-1)(q^2t^2-1)} - \frac{1}{2}\frac{q^{2g-2}t^{4g-4}(qt+1)^{2g}}{(qt^2-1)(q-1)} - \frac{1}{2}\frac{q^{2g-2}t^{4g-4}(qt-1)^{2g}}{(q+1)(qt^2+1)}$$

- $H(\mathcal{M}_{\mathrm{B}}^{d,2}; q, t) \overset{HV}{=} H_2(q,t)$
- $H_2(q,-1) = H_2(1/q,-1)q^{8g-6} = E(\mathcal{M}_{\mathrm{B}}^{d,2}; q)$
- $\frac{H_2(1,t)}{(1+t)^{2g}}$ agrees with [Hitchin 1987] for $P(\mathcal{M}_{\mathrm{B}}^{d,2}; t)$

$$P_t(\mathcal{M}) = \frac{(1+t^3)^{2g}}{(1-t^2)(1-t^4)} - \frac{t^{4g-4}(1-t)^{2g}}{4(1+t^2)} - \frac{t^{4g-3}(1+t)^{2g-2}(g-1)}{(1-t)} + \frac{t^{4g-4}(1+t)^{2g-2}(t^2-4t+1)}{4(1-t)^2}$$

$|GL_3(\mathbb{F}_q)| = (q^3-1)(q^3-q)(q^3-q^2) = q^3(q-1)^3(q+1)(q^2+q+1)$

$|T^F| = (q-1)^3$, $|T_s^F| = (q^2-1)(q-1)$, $|T_\sigma^F| = q^3-1$

$|W(T)^F| = 6$, $|W(T_s)^F| = 2$, $|W(T_\sigma)^F| = 3$

<u>Tables des caractères de $GL_3(\mathbb{F}_q)$</u>

$|C_G\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}| = q^3(q-1)^2$

$|C_G\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}^F| = q^2(q-1)$

$|Cl_G\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}| = (q^3-1)(q+1)$

$|Cl_{G^F}\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}| = (q^3-1)(q^3-q)$

		$\begin{pmatrix}a&0&0\\0&a&0\\0&0&a\end{pmatrix}$ $a\in\mathbb{F}_q^\times$	$\begin{pmatrix}a&0&0\\0&a&0\\0&0&b\end{pmatrix}$ $a\neq b,\ a,b\in\mathbb{F}_q^\times$	$\begin{pmatrix}a&0&0\\0&b&0\\0&0&c\end{pmatrix}$ a,b,c 2 à 2 $\neq$ $\in\mathbb{F}_q^\times$	$\begin{pmatrix}\lambda_1^q&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}$ $\lambda_1\in\mathbb{F}_{q^2}^\times-\mathbb{F}_q$, $\lambda_2\in\mathbb{F}_q^\times$	$\begin{pmatrix}\lambda_1&0&0\\0&\lambda_1^q&0\\0&0&\lambda_1^{q^2}\end{pmatrix}$ $\lambda_1\in\mathbb{F}_{q^3}^\times-\mathbb{F}_{q^2}$	$\begin{pmatrix}a&0&0\\0&a&1\\0&0&a\end{pmatrix}$ $a\in\mathbb{F}_q^\times$	$\begin{pmatrix}a&1&0\\0&a&1\\0&0&a\end{pmatrix}$ $a\in\mathbb{F}_q^\times$	$\begin{pmatrix}a&0&0\\0&b&1\\0&0&b\end{pmatrix}$ $a\neq b\in\mathbb{F}_q^\times$
	Nombre de classes de ce type	$q-1$	$(q-1)(q-2)$	$\frac{(q-1)(q-2)(q-3)}{6}$	$\frac{q(q-1)(q-1)}{6}$	$\frac{q(q^2-q)}{6}$	$q-1$	$q-1$	$(q-1)(q-2)$
	Cardinal des classes	1	$q^2(q^2+q+1)$	$q^3(q+1)(q^2+q+1)$	$q^2(q^2+q+1)(q^2-1)$	$(q^3-q)(q^3-q^2)$	$(q^3-1)(q+1)$	$(q^3-1)(q^3-q)$	$q^2(q^3-1)(q+1)$
6	$R_T^G(\alpha,\beta,\gamma)$ $\alpha,\beta,\gamma\in\mathrm{Irr}(\mathbb{F}_q^\times)$ (α,β,γ) 2 à 2 $\neq$	$(q+1)(q^2+q+1)\times\alpha(a)\beta(a)\gamma(a)$	$(q+1)(\alpha(a)\beta(a)\gamma(b)+\alpha(b)\beta(a)\gamma(a)+\alpha(a)\beta(b)\gamma(a))$	$\sum_{\sigma\in S_3}\alpha(\sigma a)\beta(\sigma b)\gamma(\sigma c)$	0	0	$(1+2q)\times\alpha(a)\beta(a)\gamma(a)$	$\alpha(a)\beta(a)\gamma(a)$	$\alpha(a)\beta(b)\gamma(b)+\alpha(b)\beta(a)\gamma(b)+\alpha(b)\beta(b)\gamma(a)$
1	$\mathrm{Id}_{G^F}.(\alpha\circ\det)$ $\alpha\in\mathrm{Irr}(\mathbb{F}_q^\times)$	$\alpha(a^3)$	$\alpha(a^2b)$	$\alpha(abc)$	$\alpha(\lambda_1{}^F\lambda_1\lambda_2)$	$\alpha(\lambda_1{}^F\lambda_1{}^{F^2}\lambda_1)$	$\alpha(a^3)$	$\alpha(a^3)$	$\alpha(ab^2)$
3	$\mathrm{St}_G.(\alpha\circ\det)$ $\alpha\in\mathrm{Irr}(\mathbb{F}_q^\times)$	$q^3\alpha(a^3)$	$q\alpha(a^2b)$	$\alpha(abc)$	$-\alpha(\lambda_1{}^F\lambda_1\lambda_2)$	$\alpha(\lambda_1{}^F\lambda_1{}^{F^2}\lambda_1)$	0	0	0
	$R_{T_\sigma}^G(w),\ w\neq w^q$ $w\in\mathrm{Irr}(\mathbb{F}_{q^3}^\times)$	$(q^2-1)(q-1)w(a)$	0	0	0	$w(\lambda_1)+w(\lambda_1^q)+w(\lambda_1^{q^2})$	$(1-q)w(a)$	$w(a)$	0
7	$-R_{T_s}^G(w,\alpha)$ $w\in\mathrm{Irr}(\mathbb{F}_{q^2}^\times),\neq w^q$ $\alpha\in\mathrm{Irr}(\mathbb{F}_q^\times)$	$(q^2-1)w(a)\alpha(a)$	$(q-1)w(a)\alpha(b)$	0	$-w(\lambda_1)\alpha(\lambda_2)-w(\lambda_1^q)\alpha(\lambda_2)$	0	$-w(a)\alpha(a)$	$-w(a)\alpha(a)$	$-w(b)\alpha(b)$
2	$R_\theta.(\alpha\circ\det)$ $\alpha\in\mathrm{Irr}(\mathbb{F}_q^\times)$	$q(q+1)\alpha(a^3)$	$(q+1)\alpha(a^2b)$	$2\alpha(abc)$	0	$-\alpha(\lambda_1{}^F\lambda_1{}^{F^2}\lambda_1)$	$q\alpha(a^3)$	0	$\alpha(ab^2)$
4	$R_{C_G(s)}^G(\mathrm{Id}.(\alpha\circ\det),\beta)$ $\alpha\neq\beta\in\mathrm{Irr}(\mathbb{F}_q^\times)$	$(q^2+q+1)\alpha(a^2)\beta(a)$	$\alpha(a^2)\beta(b)+(q+1)\times\alpha(a)\alpha(b)\beta(a)$	$\alpha(ab)\beta(c)+[\alpha(ac)\beta(b)+\alpha(bc)\beta(a)]$	$\alpha(\lambda_1^{q+1})\beta(\lambda_2)$	0	$(q+1)\alpha(a^2)\beta(a)$	$\alpha(a^2)\beta(a)$	$\alpha(ab)\beta(b)+\alpha(b^2)\beta(a)$
5	$R_{C_G(s)}^G(\mathrm{St}_{C_G(s)}.(\alpha\circ\det),\beta)$ $\alpha\neq\beta\in\mathrm{Irr}(\mathbb{F}_q^\times)$	$q(q^2+q+1)\times\alpha(a^2)\beta(a)$	$q\alpha(a^2)\beta(b)+(q+1)\times\alpha(a)\alpha(b)\beta(a)$	$\alpha(ab)\beta(c)+\alpha(ac)\beta(b)+\alpha(bc)\beta(a)$	$-\alpha(\lambda_1^{q+1})\beta(\lambda_2)$	0	$q\alpha(a^2)\beta(a)$	0	$\alpha(b^2)\beta(a)$

Our conjecture for $H(\mathcal{M}^{d,3}; q, t)$

$$\frac{\left(q^3t^5-1\right)^{2g}\left(q^2t^3-1\right)^{2g}}{(q^3t^6-1)(q^3t^4-1)(q^2t^4-1)(q^2t^2-1)} + \frac{q^{6g-6}t^{12g-12}\left(q^3t-1\right)^{2g}\left(q^2t-1\right)^{2g}}{(q^3t^2-1)(q^3-1)(q^2t^2-1)(q^2-1)} +$$
$$+\frac{q^{4g-4}t^{8g-8}\left(q^3t^3-1\right)^{2g}(qt-1)^{2g}}{(q^3t^4-1)(q^3t^2-1)(qt^2-1)(q-1)} + \frac{1}{3}\,\frac{q^{6g-6}t^{12g-12}\left((qt-1)^{2g}\right)^2}{(qt^2-1)^2(q-1)^2} -$$
$$-\frac{1}{3}\,\frac{q^{6g-6}t^{12g-12}\left(q^2t^2+qt+1\right)^{2g}}{(q^2t^4+qt^2+1)(q^2+q+1)} - \frac{q^{4g-4}t^{8g-8}\left(q^2t^3-1\right)^{2g}(qt-1)^{2g}}{(q^2t^4-1)(q^2t^2-1)(qt^2-1)(q-1)} -$$
$$-\frac{q^{6g-6}t^{12g-12}\left(q^2t-1\right)^{2g}(qt-1)^{2g}}{(q^2t^2-1)(q^2-1)(qt^2-1)(q-1)}.$$

Gothen's formula for $P(\mathcal{M}^{d,3}_{\mathrm{Dol}}; t)$

$$
\begin{aligned}
P_t(\mathcal{M}) =& \frac{(1+t)^{4g-4}}{(1-t)^4}\Big(2t^2 + t^4 + 2t^{2g} + 2t^{2g+2} - \tfrac{1}{4}t^{4g-4} - 3gt^{4g-3} \\
&+ (6g^2+2g-3)t^{4g-2} + (11g-12g^2)t^{4g-1} \\
&+ (6g^2-10g+\tfrac{17}{4})t^{4g} - t^{8g-6} - t^{10g-8}\Big) \\
&+ \frac{t^{2g}(1+t)^{2g-4}}{(1-t)^4(1+t^2)^2}\Big(t^{6g-8}(1+t^3)^{2g}(-2g-t^2+(2g-2)t^4) \\
&+ (1+t)^{2g}(-2t^4-2t^6+t^{2g-4}+2t^{2g-2}+t^{2g}-t^{4g-2})\Big) \\
&- \frac{2^{2g}t^{2g}(1+t)^{2g-1}}{(1-t)^4} + \frac{2gt^{8g-8}(1+t)^{2g-3}(1+t^3)^{2g-1}}{(1-t)^3(1+t^2)} \\
&+ \frac{2^{2g-1}t^{10g-8}(1+t)^{2g}}{(1-t)^3(1-t^3)} + \frac{t^{4g-4}(1-t)^{2g-1}(1+t)^{2g-1}}{4(1+t^2)} \\
&+ \frac{t^{6g-2}(1+t)^{4g-3}(1+t^2+t^4)}{(t-1)^3(1+t^2)^2(t^6-1)} + \frac{(1+t^5)^{2g}(1+t^3)^{2g-1}}{(t^2-1)(t^4-1)^2(t^3-1)}
\end{aligned}
$$

Consistency results

- [Hausel–Villegas 2008] $\rightsquigarrow$ $H(\mathcal{M}_{\mathrm{B}}^{d,2}; q,t) = H_2(q,t)$ proved by finding the weights of universal generators and extending it to a monomial basis of [Hausel–Thaddeus 2002]
- [Hausel–Villegas 2008] $\rightsquigarrow$ for $r = 3$ conjecture is consistent with [Gothen, 1994] for $P(\mathcal{M}_{\mathrm{Dol}}^{d,3}; t)$
- [Garcia-Prada–Heinloth–Schmitt 2011] $\rightsquigarrow$ for $r = 4$ conjecture is consitent with their computation $P_t(\mathcal{M}_{\mathrm{Dol}}^{d,4})$
- [Garcia-Prada–Heinloth 2012] $\rightsquigarrow$ prove Hirzebruch y-genus specialization of extended conjecture for every r
- [Chaudouard–Laumon 2012] $\rightsquigarrow$ conjecture is consistent with the nilpotent part of the Arthur-Selberg trace formula
- [Chuang–Diaconescu–Pan 2012] $\rightsquigarrow$ conjecture is equivalent with refined Gopakumar-Vafa conjecture for local curve Calabi-Yau 3-fold provided $P = W$

Sketch of approach by [Chuang–Diaconescu–Pan 2010]

- study moduli spaces of ADHM sheaves on C
- ADHM sheaf: $(E, \phi_1, \phi_2, s_1, s_2)$; a rank r vector bundle E Higgs fields $\phi_1 \in H^0(End(E))$, $\phi_2 \in H^0(End(E) \otimes K)$ sections $s_1 \in H^0(E)$, $s_2 \in H^0(E^* K)$ satisfying $[\phi_1, \phi 2] = s1 \otimes s2$
- there is a stability condition depending on $\delta \in \mathbb{R}$
- some refined Donaldson-Thomas invariants of the moduli spaces for given δ-stability condition gives polynomials $A_\delta(t)$
- $A_{\pm\infty}$ computed by (conjectural) geometric engineering
- wall-crossing formula for $A_{\delta_c+} - A_{\delta_c-}$ contains the Poincaré polynomials of lower rank Higgs moduli spaces
- $\rightsquigarrow$ recursive formula for $P_t(\mathcal{M}^{r,d}_{\mathrm{Dol}})$
- [Mozgovoy 2011] proved that the recursion is only solved by the conjectured $P_t(\mathcal{M}^{r,d}_{\mathrm{Dol}})$ provided the conjectured formula for $H(\mathcal{M}^{d,r}_{\mathrm{B}})$ is a polynomial
- this integrality is proved by [Mellit 2016]
- [Maulik-Pixton 2016$<$] announce rigorous proofs of the CDP program $\rightsquigarrow$ our conjecture for $P_t(\mathcal{M}^{r,d}_{\mathrm{Dol}})$ should be true!

A TQFT problem

- an upgraded formula $\frac{H_2(q,x,y)}{(1+qx)^g(1+qy)^g}$ gives

$$\frac{(q^2x^2y+1)^g(q^2y^2x+1)^g}{(q^2xy-1)(q^2(xy)^2-1)}+\frac{(qxy)^{2g-2}(q^2x+1)^g(q^2y+1)^g}{(q^2-1)(q^2xy-1)}-$$
$$-\frac{1}{2}\frac{(qxy)^{2g-2}(qx+1)^g(qy+1)^g}{(qxy-1)(q-1)}-\frac{1}{2}\frac{(qxy)^{2g-2}(qx-1)^g(qy-1)^g}{(q+1)(qxy+1)}$$

- it gives $H_2(1,x,y)=E(\mathcal{M}^{2,1}_{\mathrm{Dol}};x,y)$ and $E(\mathcal{M}^{2,1}_{\mathrm{Dol}};-1,y)$ is the Hirzebruch y-genus of $\mathcal{M}^{2,1}_{\mathrm{Dol}}$
- we get for $\frac{H_2(q,-1,y)}{(1-q)^g(1+qy)^g}$

$$\left((q^2y+1)(q^2y^2-1)\right)^{g-1}+\left((qy)^2(q^2-1)(q^2y+1)\right)^{g-1}-$$
$$-\frac{1}{2}\left((qy)^2(q-1)(qy+1)\right)^{g-1}-\frac{1}{2}\left((qy)^2(q-1)(qy-1)\right)^{g-1}$$

- Problem: is this the partition function of a TQFT on C?