Enumerative invariants of Higgs moduli spaces

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Plan for lectures

Lecture 1: Cohomology of Higgs moduli via $\mathbb{C}^\times$ action

Lecture 2: Cohomology of character variety via arithmetic & $P = W$

Lecture 3: Cohomology of quiver varieties and Kac's conjectures

Lecture 4: Equivariant Verlinde algebra for Higgs bundles

Ubiquity of Higgs bundles

- [Hitchin 1987] introduces Higgs bundles (E, ϕ) on Riemann surfaces as solutions of a two-dimensional reduction of the 4D Yang–Mills equations $\rightsquigarrow$ Hitchin integrable system
- [Simpson 1990] studies Higgs bundles in the framework of non-Abelian Hodge theory of a variety
- [Beilinson–Drinfeld ~1995] Geometric Langlands via quantization of Hitchin system
- [Donagi–Witten 1996] & [Moore–Nekrasov–Shatashvili 1997] importance of Higgs moduli in supersymmetric gauge theories
- [Kapustin–Witten 2005] S-duality perspective on Geometric Langlands and Higgs moduli spaces
- [Gaiotto–Neitzke–Moore 2008-] hyperkähler metric on Higgs moduli spaces maybe understood from wall-crossing
- [Ngô 2010] symmetries of Hitchin fibers $\rightsquigarrow$ fundamental lemma
- [Chuang–Diaconescu–Pan 2010-] Donaldson–Thomas theory on local curve Calabi–Yau 3-folds $\rightsquigarrow$ Higgs moduli spaces
- [Gukov-Pei 2015-] theory X on lense spaces $\rightsquigarrow$ Equivariant Verlinde formulas for Higgs moduli spaces

Moduli space of vector bundles

- C smooth complex projective curve of genus $g > 1$
- fix integers $r > 0$ and $d \in \mathbb{Z}$ always assume $(d, r) = 1$
- $\mathcal{N} :=$ moduli space of isomorphism classes of semi-stable rank r degree d vector bundles on C
- constructed using geometric invariant theory (GIT) or gauge theory
- vector bundle E is called *semi-stable* (*stable*) if every proper subbundle F satisfies

$$\mu(F) = \frac{\deg(F)}{\mathrm{rk}(F)} \overset{(<)}{\leq} \mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)}$$

- when $(d, r) = 1$ semi-stability $\Leftrightarrow$ stability $\rightsquigarrow$ $\mathcal{N}^d$ is a non-singular projective variety
- $\mathbb{E}^d$ a universal vector bundle on $\mathcal{N}^d \times C$ in the sense that $\mathbb{E}^d|_{\{E\}\times C} \cong E$

Cohomology of $\mathcal{N}$

- $H^*(\mathcal{N}^d) := H^*(\mathcal{N}^d; \mathbb{C})$ well understood
- [Harder–Narasimhan 1975] and [Atiyah–Bott 1981] obtained recursive formulae for $P_t(\mathcal{N}^d) := \sum_{i=0} \dim(H^i(\mathcal{N}^d))t^i$
- $\rightsquigarrow P_t(\mathcal{N}^d)$ depends on d, e.g. $P_t(\mathcal{N}^1) \neq P_t(\mathcal{N}^2)$ for $r = 5$
- [Atiyah–Bott 1981] $\rightsquigarrow$ the ring $H^*(\mathcal{N}^d)$ is generated by the Künneth components of $c(\mathbb{E}) \in H^*(\mathcal{N}^d) \otimes H^*(C)$
- [Jeffrey–Kirwan 1998] $\rightsquigarrow$ all intersection numbers (integrals of products of universal generators)
  as conjectured by [Witten 1992]
- [Earl–Kirwan 2005] complete set of relations among the universal generators of $H^*(\mathcal{N}^d)$
- [Teleman–Woodward 2009] Verlinde formula for $\chi(L^k) = \dim(H^0(\mathcal{N}^d; L^k))$ and generalizations
  where L is ample generator of $Pic(\mathcal{N}_d) \cong \mathbb{Z}$

Moduli space of Higgs bundles

- (E, ϕ) a Higgs bundle on C
  - E vector bundle on C
  - $\phi \in H^0(C; \mathrm{End}(E) \otimes K_C)$ *Higgs field*
- (E, ϕ) is called *semi-stable* (*stable*) if every proper ϕ-invariant subbundle $F \subset E$ satisfies

$$\mu(F) = \frac{\deg(F)}{\mathrm{rk}(F)} \overset{(<)}{\leq} \mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)}$$

- $\mathcal{M}^d :=$ moduli space of isomorphism classes of semi-stable rank r degree d Higgs bundles on C
- constructed using GIT or gauge theory
- $(d, r) = 1 \rightsquigarrow \mathcal{M}^d$ is non-singular, quasi-projective & $(\mathbb{E}, \boldsymbol{\Phi})$ universal Higgs bundle on $\mathcal{M}^d \times C$
- E stable $\Rightarrow (E, \phi)$ stable for any ϕ
  & deformation theory $\rightsquigarrow T_E\mathcal{N}_d \cong H^1(C; \mathrm{End}(E))$
  $\rightsquigarrow T^*\mathcal{N}^d \subset \mathcal{M}^d$
- $\mathbb{C}^\times$ acts on $\mathcal{M}^d$ by $(E, \phi) \mapsto (E, \lambda\phi)$; $\mathcal{M}^d$ is *semi-projective*

Semi-projective varieties

- $\mathcal{M}$ smooth quasi-projective variety *semi-projective*
    - has a $\mathbb{C}^\times$ action
    - $\mathcal{M}^{\mathbb{C}^\times}$ proper
    - $\lim_{\lambda \to 0} \lambda z$ exists for all $z \in \mathcal{M}$
- examples:
    - $\mathcal{M}^d$
    - semi-projective toric varieties
    - toric hyperkähler varieties
    - Nakajima quiver varieties
- $\mathcal{M}^{\mathbb{C}^\times} = \coprod_{i \in I} F_i$ connected components
- $U_i := \{z \in \mathcal{M} | \lim_{\lambda \to 0} \lambda z \in F_i\}$ affine bundle over F_i
- $\mathcal{M} = \coprod_{i \in I} U_i$ *Bialynicki-Birula decomposition*
- $D_i := \{z \in \mathcal{M} | \lim_{\lambda \to \infty} \lambda z \in F_i\}$ affine bundle over F_i
- $\mathcal{C} := \coprod_{i \in I} D_i \subset \mathcal{M}$ proper subvariety: *core* of $\mathcal{M}$

Cohomology of semi-projective varieties

- $H^*(\mathcal{M}) \cong \bigoplus_{i\in I} H^{*-2\operatorname{codim} U_i}(F_i)$ BB decomposition *perfect*
- $H^*_{\mathbb{C}^\times}(\mathcal{M}) \cong H^*(\mathcal{M}) \otimes H^*_{\mathbb{C}^\times}(pt)$ *equivariantly formal*
- $r : H^*_{\mathbb{C}^\times}(\mathcal{M}) \hookrightarrow H^*_{\mathbb{C}^\times}(\mathcal{M}^{\mathbb{C}^\times})$ *Kirwan injectivity*
- $\int_{\mathcal{M}} \alpha\beta := \sum_i \int_{F_i} \frac{i^*_{F_i}(\alpha\beta)}{E_{\mathbb{C}^\times}(N_{F_i})} \in H^*_{\mathbb{C}^\times}(\mathcal{M}) \otimes_{\mathbb{C}[u]} \mathbb{C}(u)$
  is a non-degenerate pairing on $H^*_{\mathbb{C}^\times}(\mathcal{M})$
- $Z := \mathcal{M}//_{\zeta_\infty}\mathbb{C}^\times$ orbifold
  $\overline{\mathcal{M}} := \mathcal{M} \times \mathbb{C}//_{\zeta_\infty}\mathbb{C}^\times = \mathcal{M} \coprod Z$ orbifold compactification
- $H^*(\overline{\mathcal{M}}) \twoheadrightarrow H^*(\mathcal{M})$ surjective
- $\mathcal{C} \subset \mathcal{M}$ is deformation retract
- $\rightsquigarrow H^*(\mathcal{M})$ has pure weight filtration

Bialiniczky-Birula decomposition for $\mathcal{M}^d$

- $[(E,\phi)] \in (\mathcal{M}^d)^{\mathbb{C}^\times}$ is fixed by $\mathbb{C}^\times$-action $\Leftrightarrow$ $(E,\phi) \cong (E,\lambda\phi)$
- $\rightsquigarrow E = E_1 \oplus \cdots \oplus E_k$

  and lower-triangular: $\phi|E_i \subset E_{i+1} \otimes K_C$ and $\phi|E_k = 0$
- its *type*: $(\mathrm{rk}E_1, \ldots, \mathrm{rk}E_k)$ ordered partition of r

  its *multi-degree* : $(\deg E_1, \ldots, \deg E_k)$ adding to d
- their locus is denoted $F^{d_1,\ldots,d_r}_{r_1,\ldots,r_k} \subset (\mathcal{M}^d)^{\mathbb{C}^\times}$

Examples for $F^{d_1,\dots,d_k}_{r_1,\dots,r_k}$

- fixed points of type $(r) \rightsquigarrow \phi = 0$ and E is stable $F^d_r \cong \mathcal{N}^d$

- fixed points of type $(1, 1, \dots, 1)$ are determined by E_1 and the divisors of zeros of $\phi_i \in H^0(E_i^{-1} \otimes E_{i+1} \otimes K_C)$

  $$\rightsquigarrow F^{d_1,\dots,d_r}_{1,\dots,1} \cong J^d(C) \times S^{d_2+2g-2-d_1}(C) \times \cdots \times S^{d_r+2g-2-d_{r-1}}(C)$$

- when $r = 2$, only two types:

  $F^d_2 \cong \mathcal{N}^d$ and

  $F^{d_1,d_2}_{1,1} \cong J^{d_1}(C) \times S^{2d_2+2g-2-d}(C)$ with stability $\rightsquigarrow d_2 < d/2$

- when $r = 3$, types: $(3), (1, 1, 1), (1, 2), (2, 1)$
  the fixed point components $F^{d_1,d_2}_{1,2}$ and $F^{d_1,d_2}_{2,1}$ can be identified with moduli spaces of Bradlow pairs
  (i.e. (V, s) where $rank(V) = 2$ and $s \in H^0(E)$)

Betti numbers for $r = 2$

- denote $P_t(X) = \sum_{i=0}^{\dim_{\mathbb{R}}(X)} \dim(H^i(X))t^i$ *Poincaré polynomial*
- [Hitchin 1987] computed Betti numbers by Morse theory

$$P_t(\mathcal{M}^1) = P_t(\mathcal{N}^1) + \sum_{i=1}^{g-1} t^{2n_i} P_t(F_{1,1}^{i,-i+1}) =$$
$$= P_t(\mathcal{N}^1) + \sum_{i=1}^{g-1} t^{2n_i} P_t\left(J^1(C) \times S^{2g-1-2i}(C)\right)$$

THEOREM (7.6). *Let $\mathcal{M}$ be the moduli space of stable pairs (V, Φ) where V is a rank-2 bundle of odd degree over a Riemann surface M of genus $g>1$, and $\Phi \in H^0(M\,; \mathrm{End}_0\, V \otimes K)$. Then*

(i) *$\mathcal{M}$ is non-compact,*
(ii) *$\mathcal{M}$ is connected and simply connected,*
(iii) *the Betti numbers b_i of $\mathcal{M}$ vanish for $i > 6g-6$,*
(iv) *the Betti numbers are given by*

$$P_t(\mathcal{M}) = \frac{(1+t^3)^{2g}}{(1-t^2)(1-t^4)} - \frac{t^{4g-4}(1-t)^{2g}}{4(1+t^2)} - \frac{t^{4g-3}(1+t)^{2g-2}(g-1)}{(1-t)} + \frac{t^{4g-4}(1+t)^{2g-2}(t^2-4t+1)}{4(1-t)^2} + \tfrac{1}{2}(2^{2g}-1)t^{4g-4}\{(1+t)^{2g-2}-(1-t)^{2g-2}\},$$

Betti numbers for $r = 3$

- [Gothen 1995] computed
$$P_t(\mathcal{M}^d) = P_t(\mathcal{N}^d) + \sum_{d_1,d_2} t^{2n_{d_1,d_2}} P_t(F_{1,2}^{d_1,d_2}) + \sum_{d_1,d_2} t^{2n_{d_1,d_2}} P_t(F_{2,1}^{d_1,d_2}) + \sum_{d_1,d_2,d_3} t^{2n} P_t(F_{1,1,1}^{d_1,d_2,d_3})$$

Theorem 3.1. *Let Σ be a closed Riemann surface of genus $g \geqslant 2$, and let Λ_0 be a holomorphic line bundle on Σ of degree d with $(d,3) = 1$. Let $\mathcal{M}$ be the moduli space of rank 3 stable Higgs bundles on Σ with fixed determinant bundle Λ_0. The Poincaré polynomial of $\mathcal{M}$ is*

$$\begin{aligned}
P_t(\mathcal{M}) = & \frac{(1+t)^{4g-4}}{(1-t)^4}\Big(2t^2 + t^4 + 2t^{2g} + 2t^{2g+2} - \tfrac{1}{4}t^{4g-4} - 3gt^{4g-3} \\
& + (6g^2 + 2g - 3)t^{4g-2} + (11g - 12g^2)t^{4g-1} \\
& + (6g^2 - 10g + \tfrac{17}{4})t^{4g} - t^{8g-6} - t^{10g-8}\Big) \\
& + \frac{t^{2g}(1+t)^{2g-4}}{(1-t)^4(1+t^2)^2}\Big(t^{6g-8}(1+t^3)^{2g}(-2g - t^2 + (2g-2)t^4) \\
& + (1+t)^{2g}(-2t^4 - 2t^6 + t^{2g-4} + 2t^{2g-2} + t^{2g} - t^{4g-2})\Big) \\
& - \frac{2^{2g}t^{2g}(1+t)^{2g-1}}{(1-t)^4} + \frac{2gt^{8g-8}(1+t)^{2g-3}(1+t^3)^{2g-1}}{(1-t)^3(1+t^2)} \\
& + \frac{2^{2g-1}t^{10g-8}(1+t)^{2g}}{(1-t)^3(1-t^3)} + \frac{t^{4g-4}(1-t)^{2g-1}(1+t)^{2g-1}}{4(1+t^2)} \\
& + \frac{t^{6g-2}(1+t)^{4g-3}(1+t^2+t^4)}{(t-1)^3(1+t^2)^2(t^6-1)} + \frac{(1+t^5)^{2g}(1+t^3)^{2g-1}}{(t^2-1)(t^4-1)^2(t^3-1)} \\
& + t^{4g-4}\big((3^{2g}-1)(1+t)^{4g-4} - 3^{2g}\big).
\end{aligned}$$

Results on cohomology of $\mathcal{M}^d$

- Betti numbers ($b_i = \dim H^i(\mathcal{M}^d)$) computed for
  - $r = 4$ [Garcia-Prada–Heinloth–Schmitt 2011]
  - $r > 4$ [Garcia-Prada–Heinloth 2012] computation converges
  - all r [Schiffmann 2015] via abs. ind. bundles (Lecture 3)
  - all r [Maulik–Pixton 2017] via DT invariants (Lecture 2)
- universal generators for cohomology ring were proved by
  - $r = 2$ [Hausel–Thaddeus 2002]
  - $r > 2$ [Markman 2002]
- relations in cohomology ring described for $r = 2$ by [Hausel–Thaddeus 2002]
- equivariant intersection numbers conjectured by [Moore–Nekrasov–Shatashvili 1997] & [Hausel–Szenes 2008 $\leq$]
- [Gukov–Pei 2015], [Andersen–Gukov–Pei 2016], [Halpern-Leistner 2016] equivariant Verlinde formulas $\chi_{\mathbb{C}^\times}(\mathcal{M}; L^k) \rightsquigarrow$ (Lecture 4)