Cohomology of hyperkähler spaces in gauge theory

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May 2005

Moment maps in various geometries
BIRS Workshop, Banff, Canada

Hyperkähler quotients

- Construction of (Hitchin-Karlhede-Lindström-Roček, 1987):
- V is a real vector space with an Euclidean pairing
- an affine space $\mathbb{M}$ modeled on the quaternionic vector space (left module) $V \times IV \times JV \times KV$, has a natural hyperkähler metric,
- G Lie group, $G \circlearrowright \mathbb{M}$ preserving the hyperkähler structure
- hyperkähler moment map:

$$\mu_{\mathbb{H}} = (\mu_I, \mu_J, \mu_K) : \mathbb{M} \to \mathbb{R}^3 \otimes \mathfrak{g}^*$$

- For $\xi \in \mathbb{R}^3 \otimes (\mathfrak{g}^*)^G$ the hyperkähler quotient

$$\mathbb{M}////_{\xi} G := \mu_{\mathbb{H}}^{-1}(\xi)/G,$$

has a natural hyperkähler metric at its smooth points

Moduli of Yang-Mills instantons on $\mathbb{R}^4$

- $P \to \mathbb{R}^4$ a $U(n)$-principal bundle over $\mathbb{R}^4$
- $\mathbb{M} =$ $\{A$ connection on $P; |\int_{\mathbb{R}^4} tr(F_A \wedge * F_A)| < \infty\}$
- $A = A_1 dx_1 + A_2 dx_2 + A_3 dx_3 + A_4 dx_4$ in a fixed gauge, where $A_i \in V = \Omega^0_{L^2}(\mathbb{R}^4, \mathrm{adP})$
- $\Rightarrow \mathbb{M}$ is modelled on $V \times V \times V \times V \Rightarrow$ has a natural hyperkähler structure
- $g \in \mathcal{G} = Ad(P)$ acts on $A \in \mathbb{M}$ by $g(A) = g^{-1}Ag + g^{-1}dg$, preserving the hyperkähler structure
- $\mu_{\mathbb{H}}(A) = 0 \Leftrightarrow F_A = *F_A$, self-dual Yang-Mills equation
- $\mathcal{M}(\mathbb{R}^4, P) = \mu_{\mathbb{H}}^{-1}(0)/\mathcal{G}$, the moduli space of finite energy self-dual Yang-Mills instantons on P, has a natural hyperkähler metric
- same story for X^4_{ALE} gravitational instanton $\Rightarrow \mathcal{M}(X^4_{ALE}, P)$ Nakajima quiver variety

Moduli space of magnetic monopoles

- Assume that A_i are independent of x_4
- $A = A_1 dx_1 + A_2 dx_2 + A_3 dx_3$ connection on $\mathbb{R}^3$
- $A_4 = \phi \in \Omega^0(\mathbb{R}^3, \mathrm{ad}P)$ the *Higgs field*
- gauge group now $\mathcal{G} = \Omega(\mathbb{R}^3, \mathrm{Ad}P)$ acts on $\mathbb{M} = \{(A, \phi)$ of finite energy $\}$ preserving the natural hyperkähler metric on $\mathbb{M}$
- $\mu_{\mathbb{H}}(A, \phi) = 0 \Leftrightarrow F_A = *d_A \phi$ Bogomolny equation
- $\mathcal{M}(\mathbb{R}^3, P) = \mu_{\mathbb{H}}^{-1}(0)/\mathcal{G}$, the moduli space of magnetic monopoles on $\mathbb{R}^3$, has a natural hyperkähler metric
- Atiyah-Hitchin 1985 finds the metric explicitly on $\mathcal{M}^2(\mathbb{R}^3, P_{SU(2)}) \Rightarrow$ describe scattering of two monopoles

Moduli space of Higgs bundles

- Assume that A_i are independent of x_3, x_4
- $A = A_1 dx_1 + A_2 dx_2$ connection on $\mathbb{R}^2$
- $\Phi = (A_3 - A_4 i)dz \in \Omega^{1,0}(\mathbb{R}^2, \mathrm{ad}P \otimes \mathbb{C})$ *complex Higgs field*
- gauge group now $\mathcal{G} = \Omega(\mathbb{R}^2, \mathrm{Ad}P)$ acts on $\mathbb{M} = \{(A, \Phi)$ of finite energy $\}$ preserving the natural hyperkähler metric on $\mathbb{M}$
- the moment map equations

$$\mu_{\mathbb{H}}(A, \Phi) = 0 \Leftrightarrow \begin{array}{c} F(A) = -[\Phi, \Phi^*], \\ d''_A \Phi = 0. \end{array}$$

  equivalent with Hitchin's self-duality equations

- replacing $\mathbb{R}^2$ with a genus g compact Riemann surface C; $\mathcal{M}(C, P) = \mu_{\mathbb{H}}^{-1}(0)/\mathcal{G}$ has a natural hyperkähler metric

Spaces diffeomorphic to $\mathcal{M}(C, P_{U(n)})$

$$\mathcal{M}^d_{\mathsf{Dol}}(GL(n,\mathbb{C})) := \left\{ \begin{array}{c} \text{moduli space of semistable rank } n \\ \text{degree } d \text{ Higgs bundles on } C \end{array} \right\}$$

$$\mathcal{M}^d_{\mathsf{DR}}(GL(n,\mathbb{C})) :=$$
$$\left\{ \begin{array}{c} \text{moduli space of flat } GL(n,\mathbb{C})\text{-connections} \\ \text{on } C \setminus \{p\}\text{, with holonomy } e^{\frac{2\pi i d}{n}} Id \text{ around } p \end{array} \right\}$$

$$\mathcal{M}^d_{\mathsf{B}}(GL(n,\mathbb{C})) := \{A_1, B_1, \ldots, A_g, B_g \in GL(n,\mathbb{C})|$$
$$A_1^{-1}B_1^{-1}A_1B_1 \ldots A_g^{-1}B_g^{-1}A_gB_g = \xi_n Id\}/GL(n,\mathbb{C})$$

L^2 harmonic forms on complete manifolds

- M complete Riemannian manifold, $\alpha \in \Omega^k(M)$ is harmonic iff $d\alpha = d * \alpha = 0$; it is L^2 iff $\int_M \alpha \wedge *\alpha < \infty$; $\mathcal{H}^*(M)$ is the space of L^2 harmonic forms

- Hodge (orthogonal) decomposition: $\Omega^*_{L^2} = \overline{d(\Omega^*_{cpt})} \oplus \mathcal{H}^* \oplus \overline{\delta(\Omega^*_{cpt})}$,

- $H^*_{cpt}(M) \to \mathcal{H}^*(M) \to H^*(M)$ is the forgetful map

- $H^*_{cpt}(M) \to H^*(M)$ is equivalent with the intersection pairing on $H^*_{cpt}(M)$, by Poincaré duality

S-duality conjectures on L^2 harmonic forms

Conjecture 1 (Sen,1994). *There are no non-trivial L^2 harmonic d forms on $\widetilde{\mathcal{M}_0^k}(\mathbb{R}^3, P_{SU(2)})$ unless $d = N/2$, when*

$$\dim(\mathcal{H}^d(\widetilde{\mathcal{M}_0^k}(\mathbb{R}^3, P_{SU(2)}))) = \phi(k)$$

Conjecture 2 (Vafa-Witten,1994). *There are no non-trivial L^2 harmonic d forms on $M = \mathcal{M}^k(X^4_{ALE}, P_{U(n)})$ unless $d = N/2$, when*

$$\dim(\mathcal{H}^d(M)) = \dim(\mathrm{im}(H^d_{cpt}(M) \to H^d(M)))$$

Results on L^2 harmonic forms

- Sen 1994, produced an L^2 harmonic form on the Atiyah-Hitchin manifold $\widetilde{\mathcal{M}_0^2}(\mathbb{R}^3, P_{SU(2)})$
- Segal-Selby 1996, proves for $M = \widetilde{\mathcal{M}_0^k}(\mathbb{R}^3, P_{SU(2)})$ $\dim(\mathrm{im}(H^{N/2}_{cpt}(M) \overset{\cong}{\to} H^{N/2}(M))) = \phi(k)$
- Hausel 1998, proves for $M = \mathcal{M}^1_{\mathrm{Dol}}(SL(2,\mathbb{C}))$ $\dim(\mathrm{im}(H^{N/2}_{cpt}(M) \to H^{N/2}(M))) = 0$
- Hitchin 2000, proves for a complete hyperkähler manifold of linear growth: $\mathcal{H}^d(M) = 0$ unless $d = N/2$; and proves Sen's conjecture for $k = 2$
- Hausel-Hunsicker-Mazzeo 2002 proves for fibered boundary manifolds M (like ALE, ALF or ALG gravitational instantons) $\mathcal{H}^{N/2}(M) = \mathrm{im}(IH^{N/2}_{\underline{m}}(\bar{M}) \to IH^{N/2}_{\bar{m}}(\bar{M}))$
- Carron 2005 proves for a QALE space M: $\mathcal{H}^{N/2}(M) = \mathrm{im}(H^{N/2}_{cpt}(\bar{M}) \to H^{N/2}(\bar{M}))$

Intersection numbers on circle compact manifolds

Definition 1. *M oriented, smooth with $U(1) \circlearrowright M$ is called* circle-compact *iff the $M^{U(1)}$ is compact. For $\alpha \in H^*_{U(1)}(M)$ we define*

$$\int_M \alpha := \sum_F \int_F \frac{i_F^*(\alpha)}{E(N_F)} \in \mathbb{Q}(u)$$

Proposition 2 (Hausel-Proudfoot 2003). *The pairing on $H^*_{U(1)}(M)$ given by*

$$\int_M \alpha \wedge \beta$$

is non-degenarate.

Intersection numbers on $\mathcal{M}^d_{Dol}$

Theorem 3 (Hausel-Szenes 2003). *Let* $\mathcal{M} := \mathcal{M}^1_{Dol}(SL(2,\mathbb{C}))$

$$BA(y) = \left(\frac{2}{u}\right)^{g-1} \frac{\left(\frac{2}{(1-y/u)^2} + u\right)^g}{\left(e^y\frac{u+y}{u-y} - e^{-y}\frac{u-y}{u+y}\right) y^{2g-2}(u^2-y^2)^{g-1}},$$

$$\begin{aligned}\int_{\mathcal{M}} e^{\alpha} &= \operatorname*{Res}_{y=0} BA(y) + \operatorname*{Res}_{y=-u} BA(y) + \operatorname*{Res}_{y=u} BA(y) \\ &= -\sum_b \operatorname*{Res}_{y=b} BA(y),\end{aligned}$$

where the sum is taken over the solutions of the Bethe-Ansatz equations:

$$e^b\frac{u+b}{u-b} = e^{-b}\frac{u-b}{u+b}$$

Corollary 4 (Hausel-Thaddeus 2000). *The Newstead relation* $\beta^g = 0$ *holds in* $H^*(\mathcal{M}^1_{Dol}(SL(2,\mathbb{C})))$ *with* $\beta \in H^4$.

Conjecture 3 (Nekrasov–Shatasvili–Moore 1998, Hausel–Szenes 2004). *The equivariant volume of* $\mathcal{M}^d_{Dol}(SL(n,\mathbb{C}))$*:*

$$\int_{\mathcal{M}} e^{\alpha} = \sum_F \int_F \frac{e^{\alpha}}{E(N_F)}$$

can be calculated as an iterated residue of a certain expression

$$BA(y_1, y_2, \ldots, y_n).$$

Some of the poles of the expression BA *are in one-to-one correspondence with ordered partitions of* n*, and the rest are the solutions of certain Bethe Ansatz equations. The iterated residue taken at a pole corresponding to the ordered partition* $n = \lambda_1 + \cdots + \lambda_k$*, agrees with the contribution to the equivariant volume by the components of type* $(\lambda_1, \lambda_2, \ldots, \lambda_k)$*. Moreover minus of the sum of the residues at the Bethe poles give the equivariant volume of the Higgs moduli space.*

Mixed Hodge Structure of Deligne

- $\bigoplus_{p,q} H^{p,q;k}(M)$ is the associated graded to the weight and Hodge filtrations on the cohomology $H^k(M,\mathbb{C})$ of a complex algebraic variety M

- $h^{p,q;k} = \dim(H^{p,q;k}(M))$, the *mixed Hodge numbers*

- $H(M;x,y,t) = \sum_{p,q,k} h^{p,q;k}(M)x^p y^q t^k$, the *mixed Hodge polynomial*

- $P(M;t) = H(M;1,1,t)$, the *Poincaré polynomial*

- $E(M;x,y) = x^n y^n H(1/x,1/y,-1)$, the *E-polynomial* of a smooth variety M.

Topological Mirror Test

Conjecture 4 (Hausel–Thaddeus 2002). *For all $d, e \in \mathbb{Z}$, satisfying $(d,n) = (e,n) = 1$, we have*

$$E_{\mathsf{st}}{}^{B^e}\left(x, y; \mathcal{M}^d_{\mathsf{DR}}(SL(n,\mathbb{C}))\right) = E_{\mathsf{st}}{}^{\hat{B}^d}\left(x, y; \mathcal{M}^e_{\mathsf{DR}}(PGL(n,\mathbb{C}))\right).$$

Conjecture 5 (Hausel–R-Villegas 2004).

$$E_{\mathsf{st}}{}^{B^e}\left(x, y, \mathcal{M}^d_{\mathsf{B}}(SL(n,\mathbb{C}))\right) = E_{\mathsf{st}}{}^{\hat{B}^d}\left(x, y, \mathcal{M}^e_{\mathsf{B}}(PGL(n,\mathbb{C}))\right).$$

Connection to Arithmetic

Theorem 5 (...,Ito 2004, Katz 2005). *If M is a smooth quasi-projective variety defined over $\mathbb{Z}$ and*

$$\#\{M(\mathbb{F}_q)\} = E(q)$$

is a polynomial in q, then

$$E(M; x, y) = E(xy).$$

Example

$$\#\{\mathbb{P}^n(\mathbb{F}_q)\} = \#\{\mathbb{P}^{n-1}(\mathbb{F}_q)\} + \#\{\mathbb{A}^n(\mathbb{F}_q)\} = q^n + q^{n-1} + \cdots + q + 1$$

$$\Downarrow$$

$$E(\mathbb{P}^n, x, y) = (xy)^n + (xy)^{n-1} + \cdots + xy + 1$$

$$\Downarrow$$

$$P(\mathbb{P}^n, t) = t^{2n} + t^{2n-2} + \cdots + t^2 + 1$$

Theorem 6 (Hausel–R-Villegas, 2004).

$$E(q) = \#\{\mathcal{M}_B(GL(n,\mathbb{F}_q))\} =$$

$$= \sum_{\chi \in Irr(GL(n,\mathbb{F}_q))} \frac{|GL(n,\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-1}} \chi(\xi_n)$$

It follows from (Hausel–Thaddeus 2000):

$$H(\mathcal{M}_B(PGL(2,\mathbb{C})); \sqrt{q}, \sqrt{q}, t) =$$
$$= \frac{(q^2t^3+1)^{2g}}{(q^2t^2-1)(q^2t^4-1)} + \frac{q^{2g-2}t^{4g-4}(q^2t+1)^{2g}}{(q^2-1)(q^2t^2-1)} -$$
$$-\frac{1}{2}\frac{q^{2g-2}t^{4g-4}(qt+1)^{2g}}{(qt^2-1)(q-1)} - \frac{1}{2}\frac{q^{2g-2}t^{4g-4}(qt-1)^{2g}}{(q+1)(qt^2+1)},$$

when $g = 3$ this equals:

$$t^{12}q^{12} + t^{12}q^{10} + 6\,t^{11}q^{10} + t^{12}q^8 + t^{10}q^{10} +$$
$$+6\,t^{11}q^8 + 16\,t^{10}q^8 + 6\,t^9q^8 + t^{10}q^6 + t^8q^8 + 26\,t^9q^6 +$$
$$+16\,t^8q^6 + 6\,t^7q^6 + t^8q^4 + t^6q^6 + 6\,t^7q^4 + 16\,t^6q^4 +$$
$$+ 6\,t^5q^4 + t^4q^4 + t^4q^2 + 6\,t^3q^2 + t^2q^2 + 1.$$

Corollary 7 (Hausel, 2005 & 2000 ⇒1998).
No pure cohomology in the middle dimension ⇒ trivial intersection form on $H^*_{cpt}(\mathcal{M}^1_B(PGL(2,\mathbb{C})))$.