Cohomology of Hyperkähler Quotients

Tamás Hausel
University of Texas at Austin

August 2004

"Moment maps in various geometries and surjectivity"

Workshop at AIM

"An electric circuit seemed to close; and a spark flashed forth the herald (as I foresaw immediately) of many long years to come of definitely directed thought and work by myself, if spared, and, at all events, on the part of others if I should even be allowed to live long enough distinctly to communicate the discovery. Nor could I resist the impulse - unphilosophical as it may have been - to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula which contains the Solution of the Problem, but, of course, the inscription has long since mouldered away."
[William Rowan Hamilton in 1843, on his invention of quaternions]

Hyperkähler quotients

- Construction of (Hitchin-Karlhede-Lindström-Roček, 1987):

- A complex Hermitian affine space,

- G Lie group, $G \circlearrowright A$ preserving the Hermitian structure

- T^*A has a natural hyperkähler metric, $G \circlearrowright T^*A$ preserving the hyperkähler structure

- we have a hyperkähler moment map:

$$\mu_{\mathbb{H}} : T^*A \to \mathbb{R}^3 \otimes \mathfrak{g}^*$$

- For $\xi \in \mathbb{R}^3 \otimes (\mathfrak{g}^*)^G$ the hyperkähler quotient

$$T^*A////_{\xi}G := \mu_{\mathbb{H}}^{-1}(\xi)/G,$$

has a natural hyperkähler metric

Examples of hyperkähler quotients

- Moduli space of Higgs bundles on a Riemann surface (Hitchin 1987); with
  - $A := \mathcal{A}$, the affine space of unitary connections on a Hermitian vector bundle V of rank n and degree d (such that $(d,n) = 1$) on the Riemann surface C
  - $G := \mathcal{G}$ gauge group of unitary automorphisms of V
  - $\mu_{\mathbb{H}} = 0$ Hitchin equation
  - the moduli space of rank n Higgs bundles of degree d on the Riemann surface C is then:

$$\mathcal{M}^d_{Dol}(GL(n,\mathbb{C})) := T^*\mathcal{A}////_0\mathcal{G}$$

- Nakajima's quiver varieties (Nakajima 1994)
  - Q quiver (oriented finite graph, with no loops), plus finite dimensional Hermitian complex vector spaces V_i and W_i, one for each of the vertices of the graph
  - $A := \bigoplus_{(i,j)\in E(Q)} Hom(V_i, V_j) \oplus$
    $\bigoplus_{i\in V(Q)} Hom(V_i, W_i)$
  - $G := U(V_1) \times U(V_2) \times ... \times U(V_k)$
  - the quiver variety is then defined:
    $$\mathcal{M}_\xi(\mathbf{v}, \mathbf{w}) := T^*A////_\xi G$$
  - examples of such manifolds are ALE gravitational instantons of (Kronheimer 1989), moduli spaces of Yang-Mills instantons on ALE spaces (Kronheimer-Nakajima 1990), and hyperpolygon spaces of (Konno 2000)

- Toric hyperkähler manifolds of (Bielawski-Dancer 1999)

  – Here $A = \mathbb{C}^n$, with the standard Hermitian structure

  – $G = U(1)^k \subset U(1)^n \circlearrowright \mathbb{C}^n$, the subgroup is given by a $k \times n$ integer matrix A.

  – The toric hyperkähler manifold then is $\mathcal{M}(A, \xi) := T^*\mathbb{C}^n ////_{\xi} G$

Cohomology of Hyperkähler Quotients

- Poincaré polynomials - Betti numbers
- Generators of the cohomology ring
- Relations for the cohomology ring

Poincaré polynomial of $T^*A////_\xi G$

- For toric hyperkähler varieties it was calculated in (Bielawski-Dancer 1999), (Hausel-Sturmfels 2002) identified it as the h-polynomial of a matroid $\Rightarrow$ combinatorics of matroids.

- For quiver varieties (Lusztig 2000) conjectures a formula $\Leftarrow$ representation theory of quantum groups. For hyperpolygon spaces (Konno 2000).

- For moduli of Higgs bundles $SL(2,\mathbb{C})$ (Hitchin 1987), $SL(3,\mathbb{C})$ (Gothen 1994). $SL(n,\mathbb{C})$ vs. $PGL(n,\mathbb{C})$ (Hausel–Thaddeus 2002) relates the Poincaré polynomial to mirror symmetry. For $GL(n,\mathbb{C})$ (Hausel–Rodriguez-Villegas 2004) conjectures a formula $\Leftarrow$ arithmetic algebraic geometry, representation theory of finite groups of Lie type.

Generators for the cohomology ring $H^*(T^*A////_\xi G)$

Conjecture 1. *The hyperkähler Kirwan map* $\kappa : H^*_G(T^*A) \cong H^*(BG) \to H^*(T^*A////_\xi G)$ *is surjective.*

- It is known for $\mathcal{M}^1_{Dol}(GL(2,\mathbb{C}))$ by (Hausel-Thaddeus 2000), for $\mathcal{M}^d_{Dol}(GL(n,\mathbb{C}))$ by (Markman 2001)

- for quiver varieties it is conjectured by (Nakajima 2002), for hyperpolygon spaces it is proven in (Konno 2000, Hausel-Proudfoot 2003)

- for toric hyperkähler manifolds $\mathcal{M}(A,\xi)$ it is known by (Konno 2000, Hausel-Sturmfels 2002)

Ring structure for $H^*(T^*A////_\xi G)$

- The ring structure of $H^*(\mathcal{M}^1_{Dol}(GL(2,\mathbb{C})))$ was explicitely calculated by (Hausel–Thaddeus 2000)

- Conjecture for the ring structure of $H^*(\mathcal{M}(\mathbf{v},\mathbf{w}))$ (Hausel 2000), for hyperpolygon spaces a complete description is in (Konno 2000, Hausel–Proudfoot 2003)

- The cohomology ring of the toric hyperkähler manifolds was determined in (Konno 2000, Hausel–Sturmfels 2002)

Integration

Definition 1. *A smooth manifold M with a circle action $U(1) \circlearrowright M$ is* circle-compact *if the set of fixed points $M^{U(1)}$ is compact.*

The natural circle action on T^*A induces a natural circle action on the hyperkähler quotients. With this circle action all of our hyperkähler manifolds $\mathcal{M}(A,\xi)$, $\mathcal{M}_\xi(\mathbf{v},\mathbf{w})$ and $\mathcal{M}^d_{Dol}(G)$ are circle compact.

Definition 2. *Let an oriented manifold M with a $U(1) \circlearrowright M$ be circle compact. Then the rationalized $U(1)$ equivariant cohomology is defined as the vector space*

$$\widehat{H}^*_{U(1)}(M) := H^*_{U(1)}(M) \otimes_{\mathbb{Q}[u]} \mathbb{Q}(u),$$

over the field $\mathbb{Q}(u)$ of rational functions. For $\alpha \in \widehat{H}^*_{U(1)}(M)$ we define

$$\int_M \alpha := \sum_F \int_F \frac{i_F^*(\alpha)}{E(N_F)} \in \mathbb{Q}(u)$$

Signature

Theorem 3 (Hausel-Proudfoot 2003). *The pairing on $\widehat{H}^*_{U(1)}(M)$ given by*

$$\int_M \alpha \wedge \beta$$

is non-degenerate.

Definition 4. *A hyperkähler manifold M is* hyper-compact *for the complex structure I, if there is a ω_I-Hamiltonian circle action on M, with proper moment map with finitely many critical points and a minimum, such that the holomorphic symplectic form $\omega_{\mathbb{C}} := \omega_J + i\omega_K$, for $\lambda \in U(1) \subset \mathbb{C}^\times$ satisfies $\lambda^*\omega_{\mathbb{C}} = \lambda\omega_{\mathbb{C}}$.*

Conjecture 2 (Hausel 2003). *Let M^{4n} be a hyper-compact hyperkähler manifold and $\sigma(M)$ denote the signature (corresponding to the ordering, given by the sign of the leading term) of the pairing on $\widehat{H}^*_{U(1)}(M)$. Then*

$$(-1)^n\sigma(M) \geq 0.$$

Abelianization

Theorem 5 (Hausel-Proudfoot 2003). *In the construction of hyperkähler quotients let* A *be finite dimensional and* G *compact. Let* $T \subset G$ *be a maximal torus of* G. *Suppose that* $T^*A////G$ *and* $T^*A////T$ *are both circle compact. If* $\alpha \in \widehat{H}^*_{U(1)\times G}(T^*A)$, *then*

$$\int_{T^*A////G} \widehat{\kappa}_G(\alpha) = \frac{1}{|W|}\int_{T^*A////T} \widehat{\kappa}_T(\alpha) \wedge e,$$

where

$$e = \prod_{a\in\Delta} a(u-a) \in (\mathrm{Sym}\,\mathfrak{t}^*)^W \otimes \mathbb{Q}[u] \cong H_{U(1)\times G}(pt).$$

Theorem 6 (Hausel–Proudfoot 2003). *Suppose that* $T^*A////G$ *and* $T^*A////T$ *are equivariantly formal, circle compact, and that the Kirwan map* $\kappa_G : H^*(T^*A) \to H^*(T^*A////G)$ *is surjective. Then*

$$H^*_{U(1)}(T^*A////G) \cong \frac{H^*_{U(1)}(T^*A////T)^W}{Ann(e)}.$$

$H^*_{U(1)}(T^*A////T)$ has been calculated in (Harada-Proudfoot 2002). Thus surjectivity of the hyperkähler Kirwan map $\Rightarrow$ description of the cohomology ring of the hyperkähler quotient. This program has been completed only in the case of the hyperpolygon spaces of Konno by (Hausel-Proudfoot 2003), obtaining the circle equivariant cohomology ring of the hyperpolygon space of (Harada–Proudfoot 2003).

Conjecture 3 (Hausel 2000). *Suppose that both $T^*A////G$ and $T^*A////T$ are hyper-compact. Then*

$$H^*(T^*A////G) \cong \frac{H^*(T^*A////T)^W}{Ann(\tilde{e})},$$

where

$$\tilde{e} = \prod_{a \in \Delta} a \in (\mathrm{Sym}\, \mathfrak{t}^*)^W \cong H_T(pt)^W.$$

Equivariant intersection numbers on $\mathcal{M}^d_{Dol}(SL(n,\mathbb{C}))$

Theorem 7 (Hausel-Szenes 2003). *Let* $\alpha \in H^2_{U(1)}(\mathcal{M}^1_{Dol}(SL(2,\mathbb{C})) \cong \mathbb{Z}$ *be the positive integral generator,* $\mathcal{M} := \mathcal{M}^1_{Dol}(SL(2,\mathbb{C}))$

$$BA(y) = \left(\frac{2}{u}\right)^{g-1} \frac{\left(\frac{2}{(1-y/u)^2} + u\right)^g}{\left(e^y \frac{u+y}{u-y} - e^{-y}\frac{u-y}{u+y}\right) y^{2g-2}(u^2-y^2)^{g-1}},$$

we have

$$\begin{aligned} \int_{\mathcal{M}} e^{\alpha} &= \operatorname*{Res}_{y=0} BA(y) + \operatorname*{Res}_{y=-u} BA(y) + \operatorname*{Res}_{y=u} BA(y) \\ &= -\sum_b \operatorname*{Res}_{y=b} BA(y), \end{aligned}$$

where the last sum is taken over the solutions of the Bethe-Ansatz equations:

$$e^b \frac{u+b}{u-b} = e^{-b}\frac{u-b}{u+b}$$

Conjecture 4 (Nekrasov–Shatasvili–Moore 1998, Hausel–Szenes 2004). *The equivariant volume of $\mathcal{M}^d_{Dol}(SL(n,\mathbb{C}))$:*

$$\int_{\mathcal{M}} e^{\alpha} = \sum_F \int_F \frac{e^{\alpha}}{E(N_F)}$$

(and indeed all equivariant intersection numbers of $\mathcal{M}^d_{Dol}(SL(n,\mathbb{C}))$), can be described as an iterated residue of a certain expression

$$BA(y_1, y_2, \ldots, y_n).$$

Some of the poles of the expression BA are in one-to-one correspondence with ordered partitions of n, and the rest are the solutions of certain Bethe Ansatz equations. The iterated residue taken at a pole corresponding to the ordered partition $n = \lambda_1 + \cdots + \lambda_k$, agrees with the contribution to the equivariant volume by the components of type $(\lambda_1, \lambda_2, \ldots, \lambda_k)$. Moreover minus of the sum of the residues at the Bethe poles give the equivariant volume of the Higgs moduli space.

Arithmetic approach

Non-Abelian Hodge theory identifies the variety $\mathcal{M}^d_{Dol}(GL(n,\mathbb{C}))$, as a smooth a manifold, with the affine hyperkähler variety, the $GL(n,\mathbb{C})$ character variety $\mathcal{M}^d_B(GL(n,\mathbb{C}))$ of a compact Riemann surface Σ of genus $g > 1$. It is defined as the space of twisted representations of the fundamental group of Σ into $GL(n,\mathbb{C})$ modulo conjugation:

$$\mathcal{M}^d_B(GL(n,\mathbb{C})) := \{A_1, B_1, \ldots, A_g, B_g \in GL(n,\mathbb{C}) | \\ A_1^{-1}B_1^{-1}A_1B_1 \ldots A_g^{-1}B_g^{-1}A_gB_g = \xi_n Id\}/GL(n,\mathbb{C})$$

The strategy of (Hausel–Rodriguez-Villegas 2003) for getting the Betti numbers of $\mathcal{M}_B^d(GL(n,\mathbb{C}))$ is to count the rational points of the variety over a finite field $\mathbb{F}_q$. Thus we have to count points of

$$\mathcal{M}_B(GL(n,\mathbb{F}_q)) := \{A_1, B_1, \ldots, A_g, B_g \in GL(n,\mathbb{F}_q)| \\ A_1^{-1}B_1^{-1}A_1B_1 \ldots A_g^{-1}B_g^{-1}A_gB_g = \xi_n Id\}/GL(n,\mathbb{F}_q),$$

for which (Frobenius–Schur 1907) gives:

$$\#\{\mathcal{M}_B(GL(n,\mathbb{F}_q))\} =$$

$$= \sum_{\chi \in Irr(GL(n,\mathbb{F}_q))} \frac{|GL(n,\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-1}} \chi(\xi_n)$$

Deligne's mixed Hodge structure for

$$M := \mathcal{M}_B^d(GL(n, \mathbb{C}))$$

gives two filtrations on the cohomology $H^k(M, \mathbb{C})$ whose associated graded is

$$\bigoplus_{p,q} H^{p,q;k}(M),$$

we denote by $h^{p,q;k}$ the dimension of $H^{p,q;k}(M)$.

$$H_n(x, y, t) := \sum_{p,q,k} h^{p,q;k}(M) x^p y^q t^k,$$

is the mixed Hodge polynomial.

Theorem 8 (Hausel–Rodriguez-Villegas 2003).

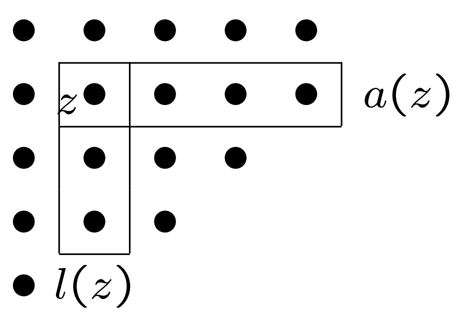
$$\begin{aligned} H_n(\sqrt{q}, \sqrt{q}, -1) &= \#\{\mathcal{M}_B(GL(n, \mathbb{F}_q))\} \\ &= \sum_{\chi \in Irr(GL(n,\mathbb{F}_q))} \frac{|GL(n, \mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-1}} \chi(\xi_n) \end{aligned}$$

Definition of $H_n(q,t)$

$$V_n(q,t) = H_n(q,t)\frac{(qt^2)^{(1-g)n(n-1)}}{(qt^2-1)(q-1)},$$

$$Z_n(q,t,T) = \exp\left(\sum_{r\geq 1} V_n(q^r, -(-t)^r)\frac{T^r}{r}\right).$$

$$\mathcal{H}_g^{\lambda}(q,t) = \prod_{x\in d(\lambda)} \frac{(qt^2)^{(2-2g)l(z)}(1+q^{h(z)}t^{2l(z)+1})^{2g}}{(1-q^{h(z)}t^{2l(z)+2})(1-q^{h(z)}t^{2l(z)})}.$$



$$\prod_{n=1}^{\infty} Z_n(q,t,T^n) = \sum_{\lambda\in\mathcal{P}} \mathcal{H}_g^{\lambda}(q,t)T^{|\lambda|}.$$

Final Conjectures

Conjecture 5 (Hausel–Rodriguez-Villegas 2004). *The mixed Hodge polynomial of* $\mathcal{M}_B^d(GL(n,\mathbb{C}))$, *is given by*

$$H_n(\sqrt{q},\sqrt{q},t) = H_n(q,t)$$

Example: $n=2$ follows from (Hausel–Thaddeus 2000):

$$H_2(\sqrt{q},\sqrt{q},t)/(qt+1)^{2g} = \\ = \frac{(q^2t^3+1)^{2g}}{(q^2t^2-1)(q^2t^4-1)} + \frac{q^{2g-2}t^{4g-4}(q^2t+1)^{2g}}{(q^2-1)(q^2t^2-1)} - \\ -\frac{1}{2}\frac{q^{2g-2}t^{4g-4}(qt+1)^{2g}}{(qt^2-1)(q-1)} - \frac{1}{2}\frac{q^{2g-2}t^{4g-4}(qt-1)^{2g}}{(q+1)(qt^2+1)},$$

when $g=3$:

$$H_2(\sqrt{q},\sqrt{q},t)/(qt+1)^6 = \\ = t^{12}q^{12} + t^{12}q^{10} + 6\,t^{11}q^{10} + t^{12}q^8 + t^{10}q^{10} + \\ +6\,t^{11}q^8 + 16\,t^{10}q^8 + 6\,t^9q^8 + t^{10}q^6 + t^8q^8 + 26\,t^9q^6 + \\ +16\,t^8q^6 + 6\,t^7q^6 + t^8q^4 + t^6q^6 + 6\,t^7q^4 + 16\,t^6q^4 + \\ +6\,t^5q^4 + t^4q^4 + t^4q^2 + 6\,t^3q^2 + t^2q^2 + 1.$$

Conjecture 6. *The Pure rings of* $\mathcal{M}_{Dol}^d(GL(n,\mathbb{C}))$ *and* $\mathcal{N}^d(GL(n,\mathbb{C}))$ *(the moduli space of rank* n *stable bundles of degree* d*), i.e. the subrings of the cohomology rings generated by the classes* $a_2, \ldots, a_n$ *are isomorphic. In particular, unlike the whole cohomology ring of* $\mathcal{N}^d(GL(n,\mathbb{C}))$*, it does not depend on* d*. Moreover the Poincaré polynomial* $PP_n(t)$ *of the pure ring is given by:*

$$PV_n(t) = PP_n(t)\frac{t^{2(1-g)n(n-1)}}{(t^2-1)},$$

$$PZ_n(t,T) = \exp\left(\sum_{r\geq 1} PV_n(t^r)\frac{T^r}{r}\right).$$

$$\mathcal{PH}_g^\lambda(t) = t^{4(1-g)n(\lambda')} \prod_{x\in d(\lambda);a(x)=0} \frac{1}{(1-t^{2h(x)})},$$

$$n(\lambda') := \sum_{z\in d(\lambda)} l(z).$$

$$\prod_{n=1}^{\infty} PZ_n(t,T^n) = \sum_{\lambda\in\mathcal{P}} \mathcal{PH}_g^\lambda(t)T^{|\lambda|}.$$

Examples:

- $n = 2$, the pure ring is generated by $\beta = a_2$, and $\beta^g = 0$, this is the famous Newstead conjecture for $\mathcal{N}^1(GL(2,\mathbb{C}))$, was first proved by (Kirwan 1992, Thaddeus 1992), while for $\mathcal{M}^1_{Dol}(GL(2,\mathbb{C}))$ it was proved in (Hausel–Thaddeus 2000)

- for $n > 2$ similar vanishings for the pure ring of $\mathcal{N}^d(GL(n,\mathbb{C}))$ was proved by (Earl–Kirwan 1999) using (Jeffrey–Kirwan 1998), which also follow from the conjecture.

$$PP_3(t) = \frac{1}{\left(t^6-1\right)\left(t^4-1\right)} + t^{12\,g-12} - \\ - \frac{t^{8\,g-8}}{t^2-1} + \frac{1}{3}\frac{t^{12\,g-12}}{\left(t^2-1\right)^2} - \frac{1}{3}\frac{t^{12\,g-12}}{t^4+t^2+1} - \\ - \frac{t^{8\,g-8}}{\left(t^4-1\right)\left(t^2-1\right)} + \frac{t^{12\,g-12}}{t^2-1}$$

"In general, although in one sense I hope that I am actually growing modest about the quaternions, from my seeing so many peeps and vistas into future expansions of their principles, I still must assert that this discovery appears to me to be as important for the middle of the nineteenth century as the discovery of fluxions was for the close of the seventeenth."
[William Rowan Hamilton in 1853, ten years after his discovery of quaternions]