Cohomology of hyperkähler moduli spaces via arithmetic harmonic analysis

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Holomorphic symplectic quotients

- $\mathbb{V}$ finite dimensional complex vector space, G complex reductive group
- representation $\rho : G \to GL(\mathbb{V})$ of G on $\mathbb{V}$, inducing $\varrho : \mathfrak{g} \to \mathfrak{gl}(\mathbb{V})$.
- inducing an action of G on $\mathbb{V} \times \mathbb{V}^*$, preserving the natural symplectic structure, with moment map: $\mu : \mathbb{V} \times \mathbb{V}^* \to \mathfrak{g}^*$ defined by $\mu(v, w)(X) = \langle \varrho(X)v, w \rangle$
- for $\xi \in (\mathfrak{g}^*)^G$ the holomorphic symplectic quotient: $\mathbb{V} \times \mathbb{V}^* ////_\xi G := (\mu^{-1}(\xi))//G$ carries a natural hyperkähler metric at its smooth points
- Examples: affine toric hyperkähler varieties when G abelian; Nakajima's quiver varieties, when ρ is constructed from a quiver, e.g. semisimple adjoint orbits in $\mathfrak{gl}(n, \mathbb{C})$

Diffeomorphic spaces in the non-Abelian Hodge theory of a curve C:

$G = GL(n)$; C is a genus g curve with generic semisimple conjugacy classes $\tilde{\mathcal{C}}_1, \ldots, \tilde{\mathcal{C}}_k \subset GL(n, \mathbb{C})$ at punctures $p_1, \ldots, p_k \in C$. The *character variety* is defined as a group-valued holomorphic symplectic quotient:

$$\mathcal{M}_{\mathsf{B}} = \{(A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_k)\, C_i \in \tilde{\mathcal{C}}_i |$$
$$A_1^{-1} B_1^{-1} A_1 B_1 \ldots A_g^{-1} B_g^{-1} A_g B_g C_1 \ldots C_k = Id\}//G(\mathbb{C})$$
$$\cong$$

$$\mathcal{M}_{\mathsf{DR}} := \left\{ \begin{array}{c} \text{moduli space of flat} \\ GL(n,\mathbb{C})\text{-connections on } C \setminus \{p_1, \ldots, p_k\} \\ \text{with holonomy around } p_i \text{ lying in } \tilde{\mathcal{C}}_i \end{array} \right\}$$

Mixed Hodge Structure of Deligne

- $\bigoplus_{p,q} H^{p,q;k}(M)$ is the associated graded to the weight and Hodge filtrations on the cohomology $H^k(M,\mathbb{C})$ of a complex algebraic variety M

- $h^{p,q;k} = \dim(H^{p,q;k}(M))$, the *mixed Hodge numbers*

- $H(M;x,y,t) = \sum_{p,q,k} h^{p,q;k}(M)x^p y^q t^k$, the *mixed Hodge polynomial*

- $P(M;t) = H(M;1,1,t)$, the *Poincaré polynomial*

- $E(M;x,y) = x^n y^n H(1/x,1/y,-1)$, the *E-polynomial* of a smooth variety M.

Arithmetic and topological content of the E-polynomial

Theorem 1 (...,Ito 2004, Katz 2005). *If M is a smooth quasi-projective variety defined over $\mathbb{Z}$ and*

$$\#\{M(\mathbb{F}_q)\} = E(q)$$

is a polynomial in q, then

$$E(M; x, y) = E(xy).$$

- MHS on $H^*(M, \mathbb{C})$ is *pure* if $h^{p,q;k} = 0$ unless $p+q = k \Leftrightarrow H(M; x, y, t) = (xyt^2)^n E(\frac{-1}{xt}, \frac{-1}{yt})$ $\Rightarrow P(M; t) = H(M; 1, 1, t) = t^{2n} E(\frac{-1}{t}, \frac{-1}{t})$; examples of varieties with pure MHS: smooth projective varieties, $\mathcal{M}_{\mathrm{Dol}}$, $\mathcal{M}_{\mathrm{DR}}$, Nakajima's quiver varieties
- in general the pure part of $H(M; x, y, t)$ is $PH(M; x, y) = \mathrm{Coeff}_{T^0}\left(H(M; xT, yT, tT^{-1})\right)$; which, for a smooth M, is always the image of the cohomology of a smooth compactification

Fourier Transform for $T^*\mathbb{C}P^n$

- Calabi's hyperkähler manifold: $T^*\mathbb{C}P^n \cong \{(v,w)\in\mathbb{C}^{n+1}\times\mathbb{C}^{n+1} | \sum_{i=1}^{n+1} v_i w_i = 1\}//GL(1)$

- $f(\xi) = q\delta_0 + (q-1)1 =$

  $$\#\{(v,w)\in\mathbb{F}_q\times\mathbb{F}_q | vw=\xi\} = \begin{cases} 2q-1 & \text{if} \quad \xi = 0 \\ q-1 & \text{if} \quad \xi \neq 0 \end{cases}$$

- $\frac{1}{q-1}\#\{(v,w)\in\mathbb{F}_q^{n+1}\times\mathbb{F}_q^{n+1} | \sum_i^n v_i w_i = 1\} =$

  $$\frac{1}{q-1} f \star f \star \cdots \star f(1) =$$

  $$\frac{q^{n/2}}{q-1}\sum_{X\in\mathbb{F}_q}\mathcal{F}(f)(X)^{n+1}\Psi(X) = \frac{q^{n/2}}{q-1}$$

  $$\sum_{X\in\mathbb{F}_q}\left(qq^{-1/2}1(X) + (q-1)q^{1/2}\delta_0(X)\right)^{n+1}\Psi(X)$$

  $$= \frac{q^{2n+1}-q^n}{q-1} = q^n(q^n + q^{n-1} + \cdots + 1)$$

- $\Rightarrow P(T^*\mathbb{C}P^n; t) = 1 + t^2 + t^4 + \cdots + t^{2n}$

Fourier Transform for $\mathcal{M}_{\mathsf{B}}$

Setup:

- $G = GL(n)$
- $C = \mathbb{P}^1$, with punctures $a_1, \ldots, a_k \in \mathbb{P}^1$
- $\tilde{\mathcal{C}}_i \subset GL(n)$ fixed semisimple conjugacy classes
- $\mathcal{M}_{\mathsf{B}} =$
  $\{(A_1, A_2, \ldots, A_k)\, A_i \in \tilde{\mathcal{C}}_i | A_1 \cdot \cdots \cdot A_k = I\}//G(\mathbb{C})$

Theorem 2 (Frobenius 1896, Hausel–R-Villegas 2004).

$$\#\{\mathcal{M}_{\mathsf{B}}(\mathbb{F}_q)\} = \sum_{\chi \in Irr(G(\mathbb{F}_q))} \frac{\chi(1)^2 |Z(G(\mathbb{F}_q))|}{|G(\mathbb{F}_q)|^2} \prod_i \frac{\chi(\tilde{\mathcal{C}}_i(\mathbb{F}_q))}{\chi(1)} |\tilde{\mathcal{C}}_i(\mathbb{F}_q)|$$

is a polynomial $E(q)$ *in* $q \Rightarrow$

$$E(\mathcal{M}_{\mathsf{B}}, x, y) = E(xy)$$

.

Example Assume $n = 3$, and all the conjugacy classes $\tilde{C}_i$ are regular semisimple:

$$E(\mathcal{M}_{\mathsf{B}};q) =$$

$$\frac{\left((q+1)\left(q^2+q+1\right)\right)^k}{\left(q^3-1\right)^2\left(q^2-1\right)^2} - \frac{\left(3\,q^2\,(q+1)\right)^k}{q^4\left(q^2-1\right)^2(q-1)^2}$$

$$+ 1/3\,\frac{\left(6\,q^3\right)^k}{q^6\,(q-1)^4} + \frac{\left(2\,q^2\left(q^2+q+1\right)\right)^k}{q^4\left(q^3-1\right)^2(q-1)^2}$$

$$+ \frac{\left(q^3\,(q+1)\left(q^2+q+1\right)\right)^k}{q^6\left(q^3-1\right)^2\left(q^2-1\right)^2} - \frac{\left(3\,q^3\,(q+1)\right)^k}{q^6\left(q^2-1\right)^2(q-1)^2}.$$

Conjecture 1 (Hausel 2004). *When* $n = 3$, $\tilde{\mathcal{C}}_i$ *are regular semisimple,* $h^{i-j}_{N-j} = h^{i+j}_{N+j}$ *for*

$$H(\mathcal{M}_{\mathsf{B}}, q, t) = \sum h^i_j q^j t^i =$$

$$\frac{\left(\left(qt^2+1\right)\left(q^2t^4+qt^2+1\right)\right)^k}{\left(q^3t^6-1\right)\left(q^3t^4-1\right)\left(q^2t^4-1\right)\left(q^2t^2-1\right)}$$

$$-\frac{\left(3\,q^2t^4\left(qt^2+1\right)\right)^k}{q^4t^8\left(q^2t^4-1\right)\left(q^2t^2-1\right)\left(qt^2-1\right)(q-1)}$$

$$+1/3\,\frac{6^k\left(qt^2\right)^{3\,k}}{q^6t^{12}\left(qt^2-1\right)^2(q-1)^2}$$

$$+\frac{\left(q^2t^4\left(2\,q^2t^2+qt^2+q+2\right)\right)^k}{q^4t^8\left(q^3t^4-1\right)\left(q^3t^2-1\right)\left(qt^2-1\right)(q-1)}$$

$$+\frac{\left(q^3t^6(q+1)\left(q^2+q+1\right)\right)^k}{q^6t^{12}\left(q^3t^2-1\right)\left(q^3-1\right)\left(q^2t^2-1\right)\left(q^2-1\right)}$$

$$-\frac{\left(3\,q^3t^6(q+1)\right)^k}{q^6t^{12}\left(q^2t^2-1\right)\left(q^2-1\right)\left(qt^2-1\right)(q-1)},$$

Conjecture 2 (Hausel 2005). *If $\mathcal{M}_{\mathsf{B}}$ is the $GL(n,\mathbb{C})$ character variety of $\mathbb{P}^1$ punctured with k generic regular semisimple conjugacy classes:*

$$H(\mathcal{M}_{\mathsf{B}};q,t) = \sum_{\substack{\lambda^1,\lambda^2,\ldots,\lambda^l \\ |\lambda^1|+\cdots+|\lambda^l|=n}} A(\lambda_1,\ldots,\lambda_l),$$

where

$$A(\lambda_1,\ldots,\lambda_l) = \frac{(r)!\left(n!(qt^2)^{\frac{n(n-1)}{2}}\prod_{i=1}^{l}\frac{1}{|\lambda^i|!}A^{\lambda^i}_{(1^n)}((qt^2)^{-1},q)\right)^k}{(-1)^{r-1}r_1!r_2!\ldots r_s!(qt^2)^{n(\lambda^1,\ldots,\lambda^l)}\prod_{i=1}^{l}c_{\lambda^i}(q,t)c'_{\lambda^i}(q,t)}.$$

Here A^{λ}_{ρ} is a linear combination of Macdonald's (q,t) Kostka polynomials, which is defined as

$$A^{\lambda}_{\rho}(u,v) = \sum_{|\mu|=n} K_{\lambda\mu}(u,v)K_{\rho\mu}.$$

The Riemann-Hilbert map

- $\mathfrak{g} = \mathfrak{gl}(n)$
- $C = \mathbb{P}^1$ with punctures $a_1, \ldots, a_k \in \mathbb{P}^n$
- $\mathcal{C}_i$ semisimple adjoint orbit in $\mathfrak{g}(\mathbb{C})$
- $Q =$ $\{(A_1, \ldots, A_k)\, A_i \in \mathcal{C}_i | A_1 + \cdots + A_k = 0\} // G(\mathbb{C})$, Nakajima's star-saped quiver variety
- Q is smooth when $\mathcal{C}_i$ are generic
- "$Q \subset \mathcal{M}_{\mathsf{DR}}$", a point in Q gives the meromorphic flat $GL(n, \mathbb{C})$-connection $\sum A_i \frac{dz}{z - a_i}$ on the trivial bundle on C.
- $\tilde{\mathcal{C}}_i = \exp(2\pi i \mathcal{C}_i) \subset G(\mathbb{C})$ is the corresponding conjugacy class
- the *Riemann-Hilbert monodromy map*

$$\nu_a : Q \to \mathcal{M}_{\mathsf{B}}$$

  is given by sending the flat connection to its holonomy.

The purity conjecture

Conjecture 3. *If $\mathcal{C}_i$ are generic, then*

$$\mu_a^* : PH^*(\mathcal{M}_{\mathsf{B}}) \xrightarrow{\cong} H^*(Q)$$

Example

- $n = 3$, $k = 3$, $\mathcal{C}_i$ regular semisimple
- Q is $\cong$ to an E_6 ALE space,
- $\mathcal{M}_{\mathsf{B}} \cong \mathcal{M}_{\mathsf{Dol}}$ an elliptic fibration with singular fibre of type $\hat{E}_6$.
- $P_t(Q) = 1 + 6t^2$
- $H(\mathcal{M}_{\mathsf{B}}; q, t) = 1 + 6qt^2 + q^2t^2$

$$\Downarrow$$

Conjecture is true in this case

Fourier transform for Q

- $\Psi : \mathbb{F}_q \to \mathbb{C}$ non-trivial additive character
- $f : \mathfrak{g}(\mathbb{F}_q) \to \mathbb{C}$ its Fourier transform $\mathcal{F}(f) : \mathfrak{g}^*(\mathbb{F}_q) \to \mathbb{C}$ at a $Y \in \mathfrak{g}^*(\mathbb{F}_q)$

$$\mathcal{F}(f)(Y) := |\mathfrak{g}(\mathbb{F}_q)|^{-1/2} \sum_{X \in \mathfrak{g}(\mathbb{F}_q)} f(X)\Psi(\langle X, Y \rangle).$$

- $\delta_{\mathcal{C}_i} : \mathfrak{g}(\mathbb{F}_q) \to \mathbb{C}$ characteristic function of $\mathcal{C}_i \subset \mathfrak{g}(\mathbb{F}_q)$

Theorem 3 (Hausel–R-Villegas 2004).

$$\#\{Q(\mathbb{F}_q)\} = \frac{|Z(G(\mathbb{F}_q))||\mathfrak{g}(\mathbb{F}_q)|^{\frac{k-2}{2}}}{|G(\mathbb{F}_q)|} \sum_{X \in \mathfrak{g}(\mathbb{F}_q)} \mathcal{F}(\delta_{\mathcal{C}_1})(X) \cdot \dots \cdot \mathcal{F}(\delta_{\mathcal{C}_k})(X)$$

Theorem 4 (Hausel–Letellier 2005). *When all $\mathcal{C}_i$ are generic regular semisimple then the pure part of the conjectured $H(\mathcal{M}_{\mathsf{B}}; q, t)$ polynomial agrees with the actual $P(Q; t)$.*

Example When $n = 3$:

$$P(Q;t) = \frac{\left(\left(t^2+1\right)\left(t^4+t^2+1\right)\right)^k}{\left(t^6-1\right)\left(t^4-1\right)} - \frac{\left(3\,t^4\left(t^2+1\right)\right)^k}{t^8\left(t^4-1\right)\left(t^2-1\right)} + 1/3\,\frac{6^k\left(t^2\right)^{3\,k}}{t^{12}\left(t^2-1\right)^2} - \frac{\left(t^4\left(t^2+2\right)\right)^k}{t^8\left(t^2-1\right)} + t^{6k-12} + \frac{(3\,t^6)^k}{t^{12}\left(t^2-1\right)}.$$

Purity conjecture for $g > 0$

Setup

- C genus g curve, punctures $a_1, \dots, a_k \in C$
- $\tilde{\mathcal{C}}_i \subset G$ semisimple conjugacy classes
- $\mathcal{M}_{\mathsf{B}} = \{(A_1, B_1, \dots, A_g, B_g, C_1, \dots, C_k)\, C_i \in \tilde{\mathcal{C}}_i |$ $A_1^{-1} B_1^{-1} A_1 B_1 \dots A_g^{-1} B_g^{-1} A_g B_g C_1 \dots C_k = Id\} // G(\mathbb{C}$
- $\mathcal{C}_i \subset \mathfrak{g}$ semisimple adjoint orbit
- $Q = \{(A_1, B_1, \dots, A_g, B_g, C_1, \dots, C_k)\, C_i \in \tilde{\mathcal{C}}_i |$

$$A_1 B_1 - B_1 A_1 + \cdots + A_g B_g - B_g A_g + C_1 + \cdots + C_k = 0\} // G(\mathbb{C})$$

Conjecture 4. *The pure part of the cohomology of $\mathcal{M}_{\mathsf{B}}$ is isomorphic with the cohomology of Q, in particular: $PH(\mathcal{M}_{\mathsf{B}}; q, t) = P(Q; t)$*

Fourier transform for $(\mathbb{C}^2)^{[n]}$

- $g = 1$ and $k = 1$, and $\mathcal{C}_1 \subset GL(n)$ is a smallest non-central semisimple orbit
- $\mathcal{M}_{\mathsf{B}} =$

  $\{(A_1, B_1, C)C \in \mathcal{C}_1 | A_1^{-1} B_1^{-1} A_1 B_1 C = Id\} // GL(n)$
- $Q \cong (\mathbb{C}^2)^{[n]}$
- Conjecture $\Rightarrow PH^*(\mathcal{M}_{\mathsf{B}}) \cong H^*((\mathbb{C}^2)^{[n]}, \mathbb{C})$.
- **[Nevins-Stafford 2003]** $\Rightarrow$

  $\mathcal{M}_{\mathsf{B}} \cong (\mathbb{C}^\times \times \mathbb{C}^\times)^{[n]}$

Conjecture 5 (Hausel 2005). *In this case:*

$$\sum_{n=1}^{\infty} H(\mathcal{M}_{\mathsf{B}}; q, t) T^n = \prod_{m=1}^{\infty} \frac{(1 + q^m t^{2m-1} T^m)^2}{(1 - q^{m-1} t^{2m-2} T^m)(1 - q^{m+1} t^{2m} T^m)}$$

Fourier transform for $\mathbb{V} \times \mathbb{V}^* ////_\xi G$

- representation $\rho : G \to GL(\mathbb{V})$ of G on $\mathbb{V}$, inducing $\varrho : \mathfrak{g} \to \mathfrak{gl}(\mathbb{V})$.

- inducing an action of G on $\mathbb{V} \times \mathbb{V}^*$, preserving the natural symplectic structure, with moment map: $\mu : \mathbb{V} \times \mathbb{V}^* \to \mathfrak{g}^*$ defined by $\mu(v,w)(X) = \langle \varrho(X)v, w\rangle$

Proposition 5 (Hausel 2005). *Let $\xi \in \mathfrak{g}(\mathbb{F}_q)^*$. The number of solutions of the equation $\mu(x) = \xi$ over the finite field $\mathbb{F}_q$ is given by the formula:*

$$\#\{(v,w) \in \mathbb{V}(\mathbb{F}_q) \times \mathbb{V}(\mathbb{F}_q)^* | \mu(v,w) = \xi\} =$$
$$|\mathfrak{g}(\mathbb{F}_q)|^{-1} |\mathbb{V}(\mathbb{F}_q)| \sum_{X \in \mathfrak{g}(\mathbb{F}_q)} C_\varrho(X) \frac{(q\delta_0(\langle X, \xi\rangle) - 1)}{q-1}$$
$$= |\mathfrak{g}(\mathbb{F}_q)|^{-1/2} |\mathbb{V}| \mathcal{F}(C_\varrho)(\xi),$$

where $C_\varrho(X) = |\ker(\varrho(X))|$.

Corollary 6. *Using the proposition one can compute the Betti numbers of toric hyperkähler varieties (recovering results of Bielawski-Dancer 2000 and Hausel-Sturmfels 2002), the Betti numbers of Hilbert schemes of n points on $\mathbb{C}^2$ (recovering results of Elingsrud-Stromme 1987, Göttsche formulas), twisted ADHM instanton moduli spaces (recovering results of Nakajima-Yoshioka 2004), and one also gets a certain generating function for the Betti numbers of all Nakajima quiver varieties.*