Cohomology of Higgs moduli spaces III

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Diffeomorphic spaces in non-Abelian Hodge theory

- C genus g curve; fix $r > 0$

$$\mathcal{M}_{\mathrm{Dol}} := \left\{ \begin{array}{c} \text{moduli space of semistable rank } r \\ \text{degree } d \text{ Higgs bundles } (E,\phi) \\ \phi \in H^0(C, \mathrm{End}(E) \otimes K) \text{ Higgs field} \end{array} \right\}$$

$$\mathcal{M}_{\mathrm{B}} := \{A_1, B_1, .., A_g, B_g \in \mathrm{GL}_r | \prod_{i=1}^{g} A_i^{-1} B_i^{-1} A_i B_i = e^{\frac{2\pi i d}{r}} Id\} /\!\!/ \mathrm{PGL}_r$$

when $(d, r) = 1$ these are smooth non-compact varieties

- Non-Abelian Hodge Theorem: $\mathcal{M}_{\mathrm{Dol}} \overset{diff}{\cong} \mathcal{M}_{\mathrm{B}}$

  [Hitchin, Donaldson, Corlette, Simpson]

- Problem: $P(\mathcal{M}_{\mathrm{Dol}}; t) = P(\mathcal{M}_{\mathrm{B}}; t)$?

Curious Hard Lefschetz

- recall [Hausel-Villegas 2008] calculates $E(\mathcal{M}_{\mathrm{B}};q)=|\mathcal{M}_B(\mathbb{F}_q)|=\sum_{\chi\in Irr(\mathrm{GL}_r(\mathbb{F}_q))}\frac{|\mathrm{GL}_r(\mathbb{F}_q)|^{2g-2}(q-1)}{\chi(1)^{2g-2}}\chi(\xi_r)$
- we find $E(\mathcal{M}_{\mathrm{B}};1/q)=q^{-D_r}E(\mathcal{M}_B;q)$ palindromic by *Alvis-Curtis duality*

$$q^{\frac{r(r-1)}{2}}\chi(1)(1/q)=\chi'(1)(q) \text{ for dual pair } \chi,\chi'\in Irr(\mathrm{GL}_r(\mathbb{F}_q))$$

Conjecture (Hausel–Villegas 2008)

$$\begin{array}{cccc} L^l: & Gr^W_{d-2l}(H^{i-l}(\mathcal{M}_{\mathrm{B}})) & \stackrel{\cong}{\to} & Gr^W_{d+2l}H^{i+l}(\mathcal{M}_{\mathrm{B}}) \\ & x & \mapsto & x\cup\alpha^l \end{array}$$

where $\alpha\in W_4H^2(\mathcal{M}_{\mathrm{B}})$

- theorem of [Hausel–Villegas 2008] for $r=2$
- implied functional equation on the conjectured $H(\mathcal{M}_{\mathrm{B}};q,t)=(qt)^{D_r}H(\mathcal{M}_{\mathrm{B}};\frac{1}{qt^2},t)$ holds

Perverse filtration

- $f : X \to Y$ a *proper* map between complex algebraic varieties of relative dimension d
- (de Cataldo-Migliorini 2005) introduce *perverse filtration* $\subset P_i \subset P_{i+1} \subset \dots P_k(X) \cong H^k(X)$ from the study of the Beilinson-Bernstein-Deligne-Gabber decomposition theorem for $Rf_*(\mathbb{Q}_X)$ into perverse sheaves
- the Relative Hard Lefschetz Theorem holds:

$$\begin{array}{rccc} L^l : & Gr^P_{d-l}(H^*(X)) & \to & Gr^P_{d+l}H^{*+2l}(X) \\ & x & \mapsto & x \cup \alpha^l \end{array}$$

where $\alpha \in H^2(X)$ is a relative ample class

Hitchin system

- Hitchin map $\chi : \begin{array}{ccc} \mathcal{M}_{\mathrm{Dol}} & \to & \mathbb{A} \cong \bigoplus_{i=1}^{n} H^0(C; K_C^i) \\ (E,\phi) & \mapsto & \mathrm{charpol}(\phi) \end{array}$
- when $r = 1$: $(L,\phi) \in \mathcal{M}_{\mathrm{Dol}}^{d;1} \cong J^d(C) \times H^0(C; K_C)$
  Hitchin map is projection to second factor:
  $\chi : J^d(C) \times H^0(C; K_C) \to H^0(C; K_C)$
- Serre duality defines symplectic form on
  $T_{[E,\phi]} \cong \mathbb{H}^1(C; \mathrm{End}(E) \overset{[\phi,.]}{\to} \mathrm{End}(E) \otimes K_C)$
  $\rightsquigarrow \mathcal{M}_{\mathrm{Dol}}$ naturally symplectic

Theorem (Hitchin 1987)

χ is completely integrable Hamiltonian system

Theorem (Hitchin 1987, Nitsure 1991, Faltings 1993)

χ is proper

- $\rightsquigarrow$ generic fiber is an Abelian variety
- $\mathcal{M}_{\mathrm{Dol}}$ is semi-projective $\chi^{-1}(0) = \mathcal{C} = \coprod_i D_i$ nilpotent cone

Fibers of the Hitchin map

- $\pi_* \mathcal{O}_{T^*C} = \mathrm{Sym}^*(TC)$
- quasi-coherent sheaves on $T^*C \leftrightsquigarrow$ quasi-coherent sheaves on C with an action of $Sym^*(TC)$
- pure dimension 1 sheaves on T^*C with proper support $\leftrightsquigarrow$ Higgs bundles on C
- support is subscheme of the spectral curve of the Higgs bundle $s^{-1}(0) \subset T^*C$ corresponding to the eigenvalues of ϕ $s := \lambda^n + a_1\lambda^{n-1} + \cdots + a_n \in H^0(X, \pi^* K^n)$
- Higgs bundles in the same fiber $\chi^{-1}(a)$ have the same spectral curve $C_a \subset T^*C$ given by $a \in \mathbb{A}$
- $\chi^{-1}(a)$ is the moduli space of stable rank 1 torsion-free sheaves on C_a , possibilities:
  - $\mathbb{A}_{reg} \subset \mathbb{A}$ where C_a smooth; $\chi^{-1}(a) \cong J(C_a)$
  - $\mathbb{A}_{int} \subset \mathbb{A}$ where C_a integral; $\chi^{-1}(a) \cong \overline{J(C_a)}$
  - $\mathbb{A}_{red} \subset \mathbb{A}$ where C_a is reduced; $\chi^{-1}(0) \cong \overline{J_\xi(C_a)}$

Main conjecture

Conjecture ("P=W", de Cataldo-Hausel-Migliorini 2010)

$P_k(\mathcal{M}_{\mathrm{Dol}}) \cong W_{2k}(\mathcal{M}_{\mathrm{B}})$ under the isomorphism $H^*(\mathcal{M}_{\mathrm{Dol}}) \cong H^*(\mathcal{M}_{\mathrm{B}})$ from non-Abelian Hodge theory

- recipe (de Cataldo-Migliorini, 2008) for perverse filtration when X smooth and Y affine:
  take $Y_0 \subset \cdots \subset Y_i \subset \ldots Y_d = Y$
  s.t. Y_i generic with $\dim(Y_i) = i$ then

$$P_{k-i-1}H^k(X) = \mathrm{ker}(\mathrm{H^k(X) \to H^k(f^{-1}(Y_i))})$$

- thus Conjecture $\Rightarrow$ "topology of Hitchin map reflects the arithmetic of the character variety"

Universal Classes

- now on let $r = 2$, i.e. study $\mathrm{GL}(2)$ Higgs bundles
- $\mathbb{E} \to \mathcal{M}_{\mathrm{Dol}} \times C$ and $\Phi : \mathbb{E} \to \mathbb{E} \otimes K_C$, universal Higgs bundle $(\mathbb{E}, \Phi)|_{(E,\phi)\times C} = (E, \phi)$
- $$c_2(End(\mathbb{E})) = 2\alpha[C]^* + \sum_{i=1}^{2g} 4\psi_i e_i - \beta$$

  for some $\alpha \in H^2(\mathcal{M}_{\mathrm{Dol}})$, $\psi_i \in H^3(\mathcal{M}_{\mathrm{Dol}})$ and $\beta \in H^4(\mathcal{M}_{\mathrm{Dol}})$. Generate $H^*(\mathcal{M}_{\mathrm{Dol}}(\mathrm{PGL}_2))$. (Hausel-Thaddeus 2002)
- (Hausel-Villegas 2008) $\Rightarrow \alpha, \psi_i, \beta \in W_4$,
- Conjecture $\Rightarrow \alpha, \psi_i, \beta \in P_2 \Rightarrow$ $\psi_i, \beta \in \ker(\mathrm{H}^*(\mathcal{M}_{\mathrm{Dol}}) \to \mathrm{H}^*(\chi^{-1}(\mathrm{Y}_0)))$
- Yes! was proved by (Thaddeus 1990)
- $\beta \in P_2 H^4(\mathcal{M}_{\mathrm{Dol}})$ would mean $\beta \in \ker(\mathrm{H}^4(\mathcal{M}_{\mathrm{Dol}}) \to \mathrm{H}^4(\chi^{-1}(\mathrm{Y}_1)))$ i.e. β vanishes over a generic curve in $\mathbb{A}$.

Applications of Ngô's support theorem

Theorem (Ngô, 2008)

$\chi^{int} : \mathcal{M}_{int} \subset \mathcal{M}_{\mathrm{Dol}} \to \mathbb{A}_{int} \subset \mathbb{A}$ over points with integral spectral curve.

$$R\chi^{int}_{*}\underline{\mathbb{Q}} \simeq \bigoplus_{i \geq 0} \mathcal{IC}_{\mathbb{A}_{int}}(L^{\Lambda i})[-i],$$

where $L^{\Lambda i} = R^i\chi^{int}_{*}(\underline{\mathbb{Q}}) = \Lambda^i R^1\chi^{int}_{*}(\underline{\mathbb{Q}})$ on $\mathbb{A}_{reg}$, where spectral curve is smooth.

Applications for $r = 2$:

- $(2-2g)\beta = c_2(\mathcal{M}_{\mathrm{Dol}})$ vanishes on $\mathbb{A}_{reg}$. Ngô's support theorem $\Rightarrow \beta$ vanishes on the generic line $\Rightarrow \beta \in P_2$.
- By computation $\mathcal{IC}_{\mathbb{A}_{int}}(L^{\Lambda i}) = j^{reg}_{*}(L^{\Lambda i})$, i.e. no higher cohomology sheaves $\Rightarrow$ perverse filtration on $H^*(\mathcal{M}_{int})$ is compatible with cup-product
  $\Rightarrow P \subset W$ on $H^*(\mathcal{M}_{\mathrm{Dol}})$ (almost)
  $\Rightarrow$ CHL (Thm for $r = 2$) and RHL imply $P = W$ (almost)

$P = W$ when $n = 2$

- One more ingredient needed: the intersection form $H_c^*(\mathcal{M}_{\mathrm{Dol}}) \to H^*(\mathcal{M}_{\mathrm{Dol}})$ is trivial when $r = 2$ by (Hausel, 1998)

Theorem (de Cataldo, Hausel, Migliorini 2009)

$P_k(H^*(\mathcal{M}_{\mathrm{Dol}})) \cong W_{2k}(H^*(\mathcal{M}_{\mathrm{B}}))$ when $r = 2$.

- Thus in particular $\beta^i \in P_{2i}(H^{4i}(\mathcal{M}_{\mathrm{Dol}}))$ i.e. vanishes over a generic $2i - 1$ dimensional subvariety in $\mathbb{A}$
- the pure subring $\langle 1, \beta, \ldots, \beta^{g-1} \rangle$ is dual by RHL with the g-dimensional $H^{mid}(\mathcal{M}_{\mathrm{Dol}})$

Weak Hard Lefschetz

- relative Hard Lefscetz $\Rightarrow$ weak Hard Lefschetz

  $L^k : H^{D_r/2-k}(\mathcal{M}_{\mathrm{Dol}}) \hookrightarrow H^{D_r/2+k}(\mathcal{M}_{\mathrm{Dol}})$ in particular

  $L : H^i(\mathcal{M}_{\mathrm{Dol}}) \hookrightarrow H^{i+2}(\mathcal{M}_{\mathrm{Dol}})$ when $i < D_r/2$
- this holds more generally for semi-projective varieties $\mathcal{M}$ with a symplectic form of homogeneity 1 $\rightsquigarrow$ core is Lagrangian
- examples include
  - $\mathcal{M}_{\mathrm{Dol}}$
  - Nakajima quiver varieties
  - toric hyperkähler varieties
    $\rightsquigarrow$ new restrictions on face vectors of matroid complexes
- what do Poincaré polynomials of such semi-projective varieties look like?