

Cohomology of Higgs moduli spaces II

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- C genus g curve; fix $r > 0$

$$\mathcal{M}_{\text{Dol}}^{d,r} := \left\{ \begin{array}{l} \text{moduli space of semistable rank } r \\ \text{degree } d \text{ Higgs bundles } (E, \phi) \\ \phi \in H^0(C, \text{End}(E) \otimes K) \text{ Higgs field} \end{array} \right\}$$

$$\mathcal{M}_{\text{B}}^{d,r} := \{A_1, B_1, \dots, A_g, B_g \in \text{GL}_r \mid \prod_{i=1}^g A_i^{-1} B_i^{-1} A_i B_i = e^{\frac{2\pi i d}{r}} \text{Id}\} // \text{PGL}_r$$

when $(d, r) = 1$ these are smooth non-compact varieties

- Non-Abelian Hodge Theorem: $\mathcal{M}_{\text{Dol}}^{d,r} \stackrel{\text{diff}}{\cong} \mathcal{M}_{\text{B}}^{d,r}$
[Hitchin, Donaldson, Corlette, Simpson]
- Problem: $P(\mathcal{M}_{\text{Dol}}^{d,r}; t) = P(\mathcal{M}_{\text{B}}^{d,r}; t)$?

- when $r = 1 \rightsquigarrow$

$$\phi : H^0(C; \text{End}(E) \otimes K_C) \cong H^0(C; K_C) \rightsquigarrow$$

$$\mathcal{M}_{\text{Dol}}^{d,1} \cong T^* J^d(C) \cong J^d(C) \times H^0(C; K_C)$$

\cong_{diff}

$$\mathcal{M}_{\text{B}}^{d,1} \cong \text{Hom}(\pi_1(C), \text{GL}_1(\mathbb{C})) \cong (\mathbb{C}^\times)^{2g}$$

- $r > 1$ and $g = 0$ both $\mathcal{M}_{\text{B}}^{d,r}$ and $\mathcal{M}_{\text{Dol}}^{d,r}$ are empty
- $r > 1$ and $g = 1 \rightsquigarrow$ Stone-von Neumann \rightsquigarrow

$$\mathcal{M}_{\text{B}}^{d,r} \cong (\mathbb{C}^*)^2$$

\cong_{diff}

$$\mathcal{M}_{\text{Dol}}^{d,r} \cong T^* J^d(C) \cong T^* C \cong C \times H^0(C; K_C) \cong C \times \mathbb{C}$$

Mixed Hodge polynomials

- (Deligne 1971) proved the existence of $W_0 \subset \cdots \subset W_i \subset \cdots \subset W_{2k} = H^k(X; \mathbb{Q})$ for any complex algebraic variety X which is
 - functorial
 - compatible with cup-product
- $H(X; q, t) = \sum \dim(W_i/W_{i-1}(H^k(X))) t^k q^{\frac{i}{2}}$, *mixed Hodge polynomial*
- $P(X; t) = H(X; 1, t)$, *Poincaré polynomial*
- $E(X; q) = q^D H(X; 1/q, -1)$, *E-polynomial of X .*

Theorem (Katz 2008)

If M is a smooth quasi-projective variety defined over \mathbb{Z} and

$$\#\{M(\mathbb{F}_q)\} = E(q)$$

is a polynomial in q , then $E(M; q) = E(q)$.

Examples for mixed Hodge polynomials

- M smooth semi-projective variety \rightsquigarrow
 $W_{k-1}(H^k(M)) = 0$ because it is smooth
 $W_k(H^k(M)) = W_k(H^k(C)) = H^k(C) = H^k(M) \leftarrow C$ projective
such cohomology is called *pure*
 $\rightsquigarrow H(M; q, t) = P(M; tq^{\frac{1}{2}})$
 $\rightsquigarrow E(M; q) = q^D P(M; -q^{-\frac{1}{2}})$
- $H(T^*J^d(C); q, t) = H(J^d(C); q, t) = (1 + q^{\frac{1}{2}}t)^{2g}$ MHS is pure
 $E(T^*J^d(C); q) = q^g(1 - q^{\frac{1}{2}})^{2g}$ not polynomial count
- $\#(\mathrm{GL}_1(\mathbb{F}_q)) = q - 1 \stackrel{\text{Katz}}{\Rightarrow} E(\mathrm{GL}_1(\mathbb{C}); q) = E(\mathbb{C}^\times; q) = q - 1 \rightsquigarrow$
 $W_1 H^1(\mathbb{C}^\times) = 0 \rightsquigarrow H(\mathbb{C}^\times; q, t) = 1 + qt$ MHS *not pure*
- $\rightsquigarrow H((\mathbb{C}^\times)^{2g}; q, t) = (1 + qt)^{2g}$ and $E((\mathbb{C}^\times)^{2g}; q) = (q - 1)^{2g}$
- $E((\mathbb{C}^\times)^{2g}; q)$ palindromic $\Leftrightarrow E((\mathbb{C}^\times)^{2g}; q) = q^{2g} E((\mathbb{C}^\times)^{2g}; 1/q)$
- extends to $H(\mathbb{C}^\times)^{2g}; q, t) = (qt)^{2g} H(\mathbb{C}^\times)^{2g}; 1/qt^2, t)$

Theorem (Frobenius 1896, Hurwitz 1902, Freed-Quinn 1993,...)

Let $z \in G$ in a finite group G then

$$\begin{aligned} \#\{a_j, b_j \in G \mid [a_1, b_1] \dots [a_g, b_g] = z\} &= \\ &= \sum_{\chi \in \text{Irr}(G)} \frac{|G|^{2g-1} \chi(z)}{\chi(1)^{2g-2} \chi(1)} \end{aligned}$$

- (Hausel-Villegas 2008) calculates

$$E(\mathcal{M}_B^{d,r}; q) \stackrel{\text{Katz}}{=} |\mathcal{M}_B^{d,r}(\mathbb{F}_q)| = \sum_{\chi \in \text{Irr}(\text{GL}_r(\mathbb{F}_q))} \frac{|\text{GL}_r(\mathbb{F}_q)|^{2g-2} (q-1) \chi(\xi_r^d)}{\chi(1)^{2g-2} \chi(1)}$$

- $\leadsto E(\mathcal{M}_B^{d,r}; q) = E(\mathcal{M}_B^{d',r}; q)$ when $(d, r) = (d', r) = 1$
- $\mathcal{M}_B^{d,r}$ and $\mathcal{M}_B^{d',r}$ Galois conjugate \Rightarrow
 $H(\mathcal{M}_B^{d,r}; q, t) = H(\mathcal{M}_B^{d',r}; q, t)$

Character table of $\mathrm{GL}_2(\mathbb{F}_q)$

Table 1: characters of $\mathbf{GL}_2(\mathbb{F}_q)$
 (note that $|\mathbf{GL}_2(\mathbb{F}_q)| = q(q-1)^2(q+1)$)

Classes	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ $a \in \mathbb{F}_q^\times$	$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ $a, b \in \mathbb{F}_q^\times$ $a \neq b$	$\begin{pmatrix} x & 0 \\ 0 & {}^F x \end{pmatrix}$ $x \in \mathbb{F}_{q^2}^\times$ $x \neq {}^F x$	$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ $a \in \mathbb{F}_q^\times$
Number of classes of this type	$q-1$	$\frac{(q-1)(q-2)}{2}$	$\frac{q(q-1)}{2}$	$q-1$
Cardinal of the class	1	$q(q+1)$	$q(q-1)$	q^2-1
$R_{\mathbf{T}}^G(\alpha, \beta)$ $\alpha, \beta \in \mathrm{Irr}(\mathbb{F}_q^\times)$ $\alpha \neq \beta$	$(q+1)\alpha(a)\beta(a)$	$\alpha(a)\beta(b) + \alpha(b)\beta(a)$	0	$\alpha(a)\beta(a)$
$-R_{\mathbf{T}_s}^G(\omega)$ $\omega \in \mathrm{Irr}(\mathbb{F}_{q^2}^\times)$ $\omega \neq \omega^q$	$(q-1)\omega(a)$	0	$-\omega(x) - \omega({}^F x)$	$-\omega(a)$
$\mathrm{Id}_G.(\alpha \circ \det)$ $\alpha \in \mathrm{Irr}(\mathbb{F}_q^\times)$	$\alpha(a^2)$	$\alpha(ab)$	$\alpha(x \cdot {}^F x)$	$\alpha(a^2)$
$\mathrm{St}_G.(\alpha \circ \det)$ $\alpha \in \mathrm{Irr}(\mathbb{F}_q^\times)$	$q\alpha(a^2)$	$\alpha(ab)$	$-\alpha(x \cdot {}^F x)$	0

Example $GL_2(\mathbb{F}_q)$



$$\frac{1}{|\mathrm{PGL}_2(\mathbb{F}_q)|(q-1)^{2g}} \#\{a_j, b_j \in \mathrm{GL}_2(\mathbb{F}_q) \mid [a_1, b_1] \dots [a_g, b_g] = -1\} =$$

$$\frac{1}{|\mathrm{PGL}_2(\mathbb{F}_q)|(q-1)^{2g}} \sum_{\chi \in \mathrm{Irr}(\mathrm{GL}_2(\mathbb{F}_q))} \frac{|\mathrm{GL}_2(\mathbb{F}_q)|^{2g-1} \chi(-1)}{\chi(1)^{2g-2} \chi(1)} =$$

$$(q^2-1)^{2g-2} + q^{2g-2}(q^2-1)^{2g-2} - \frac{1}{2}q^{2g-2}(q-1)^{2g-2} - \frac{1}{2}q^{2g-2}(q+1)^{2g-2}$$

$$\stackrel{\text{Katz}}{=} \frac{E(\mathcal{M}_B^{d,2})}{(q-1)^{2g}}.$$

- e.g. $g = 0$ gives 0 when $g = 1$ it gives 1

Conjecture (Hausel-Villegas, 2008)

$$\sum_{\lambda} \prod \frac{(z^{2l+1}-w^{2a+1})^{2g}}{(z^{2l+2}-w^{2a})(z^{2l}-w^{2a+2})} T^{|\lambda|} = \exp \left(\sum_{r,k} \frac{H(\mathcal{M}_B^{d,r}; w^{2k}, -(zw)^{-2k})(zw)^{Dr}}{(z^{2k}-1)(1-w^{2k})} \frac{T^{rk}}{k} \right)$$

- when $g = 0$ $\mathcal{M}_B^{d,r}$ empty unless $r = 1$ when $\mathcal{M}_B^{d,1} = pt \stackrel{HV}{\sim}$

$$\sum_{\lambda} \prod \frac{T^{|\lambda|}}{(z^{2l+2}-w^{2a})(z^{2l}-w^{2a+2})} = \exp \left(\sum_{k \geq 1} \frac{1}{(z^{2k}-1)(1-w^{2k})} \frac{T^k}{k} \right)$$

proved by [Garsia–Haiman, 1996]

- when $g = 1$ $\mathcal{M}_B^{d,r} = (\mathbb{C}^*)^2$ by Stone-von Neumann $\stackrel{HV}{\sim}$

$$\sum_{\lambda} \prod \frac{(z^{2l+1}-w^{2a+1})^2}{(z^{2l+2}-w^{2a})(z^{2l}-w^{2a+2})} T^{|\lambda|} = \exp \left(\sum_{k \geq 1} \frac{(z^k-w^k)^2}{(z^{2k}-1)(1-w^{2k})(1-T^k)} \frac{T^k}{k} \right)$$

proved geometrically by [Waelder, 2008]

Formula for $H(\mathcal{M}_B^{d,2}; q, t)$

- when $r = 2$ Conjecture \leadsto

$$\begin{aligned} \frac{H_2(q, t)}{(1 + qt)^{2g}} &= \frac{(q^2 t^3 + 1)^{2g}}{(q^2 t^2 - 1)(q^2 t^4 - 1)} + \frac{q^{2g-2} t^{4g-4} (q^2 t + 1)^{2g}}{(q^2 - 1)(q^2 t^2 - 1)} - \\ &- \frac{1}{2} \frac{q^{2g-2} t^{4g-4} (qt + 1)^{2g}}{(qt^2 - 1)(q - 1)} - \frac{1}{2} \frac{q^{2g-2} t^{4g-4} (qt - 1)^{2g}}{(q + 1)(qt^2 + 1)} \end{aligned}$$

- $H(\mathcal{M}_B^{d,2}; q, t) \stackrel{HV}{=} H_2(q, t)$
- $H_2(q, -1) = H_2(1/q, -1) q^{8g-6} = E(\mathcal{M}_B^{d,2}; q)$
- $\frac{H_2(1, t)}{(1+t)^{2g}}$ agrees with [Hitchin 1987] for $P(\mathcal{M}_B^{d,2}; t)$

$$\begin{aligned} P_t(\mathcal{M}) &= \frac{(1 + t^3)^{2g}}{(1 - t^2)(1 - t^4)} - \frac{t^{4g-4} (1 - t)^{2g}}{4(1 + t^2)} \\ &- \frac{t^{4g-3} (1 + t)^{2g-2} (g - 1)}{(1 - t)} + \frac{t^{4g-4} (1 + t)^{2g-2} (t^2 - 4t + 1)}{4(1 - t)^2} \end{aligned}$$

- [Hausel–Villegas 2008] $\rightsquigarrow H(\mathcal{M}_B^{d,2}; q, t) = H_2(q, t)$
proved by finding the weights of universal generators and
extending it to a monomial basis of [Hausel–Thaddeus 2002]
- [Hausel–Villegas 2008] \rightsquigarrow for $r = 3$ conjecture is consistent
with [Gothen, 1994] for $P(\mathcal{M}_{\text{Dol}}^{d,3}; t)$
- [Garcia-Prada–Heinloth–Schmitt 2011] \rightsquigarrow for $r = 4$
conjecture is consistent with their computation $P_t(\mathcal{M}_{\text{Dol}}^{d,4})$
- [Garcia-Prada–Heinloth 2012] \rightsquigarrow prove Hirzebruch y -genus
specialization of extended conjecture for every r
- [Chuang–Diaconescu–Pan 2012] \rightsquigarrow conjecture is equivalent
with refined Gopakumar–Vafa conjecture for local curve
Calabi–Yau 3-fold provided $P = W$

Our conjecture for $H(\mathcal{M}^{d,3}; q, t)$

$$\begin{aligned} & \frac{(q^3 t^5 - 1)^{2g} (q^2 t^3 - 1)^{2g}}{(q^3 t^6 - 1)(q^3 t^4 - 1)(q^2 t^4 - 1)(q^2 t^2 - 1)} + \frac{q^{6g-6} t^{12g-12} (q^3 t - 1)^{2g} (q^2 t - 1)^{2g}}{(q^3 t^2 - 1)(q^3 - 1)(q^2 t^2 - 1)(q^2 - 1)} + \\ & + \frac{q^{4g-4} t^{8g-8} (q^3 t^3 - 1)^{2g} (qt - 1)^{2g}}{(q^3 t^4 - 1)(q^3 t^2 - 1)(qt^2 - 1)(q - 1)} + \frac{1}{3} \frac{q^{6g-6} t^{12g-12} ((qt - 1)^{2g})^2}{(qt^2 - 1)^2 (q - 1)^2} - \\ & - \frac{1}{3} \frac{q^{6g-6} t^{12g-12} (q^2 t^2 + qt + 1)^{2g}}{(q^2 t^4 + qt^2 + 1)(q^2 + q + 1)} - \frac{q^{4g-4} t^{8g-8} (q^2 t^3 - 1)^{2g} (qt - 1)^{2g}}{(q^2 t^4 - 1)(q^2 t^2 - 1)(qt^2 - 1)(q - 1)} - \\ & - \frac{q^{6g-6} t^{12g-12} (q^2 t - 1)^{2g} (qt - 1)^{2g}}{(q^2 t^2 - 1)(q^2 - 1)(qt^2 - 1)(q - 1)}. \end{aligned}$$

Gothen's formula for $P(\mathcal{M}_{\text{Dol}}^{d,3}; t)$

$$\begin{aligned}
 P_t(\mathcal{M}) = & \frac{(1+t)^{4g-4}}{(1-t)^4} \left(2t^2 + t^4 + 2t^{2g} + 2t^{2g+2} - \frac{1}{4}t^{4g-4} - 3gt^{4g-3} \right. \\
 & + (6g^2 + 2g - 3)t^{4g-2} + (11g - 12g^2)t^{4g-1} \\
 & \left. + (6g^2 - 10g + \frac{17}{4})t^{4g} - t^{8g-6} - t^{10g-8} \right) \\
 & + \frac{t^{2g}(1+t)^{2g-4}}{(1-t)^4(1+t^2)^2} \left(t^{6g-8}(1+t^3)^{2g}(-2g - t^2 + (2g-2)t^4) \right. \\
 & \left. + (1+t)^{2g}(-2t^4 - 2t^6 + t^{2g-4} + 2t^{2g-2} + t^{2g} - t^{4g-2}) \right) \\
 & - \frac{2^{2g}t^{2g}(1+t)^{2g-1}}{(1-t)^4} + \frac{2gt^{8g-8}(1+t)^{2g-3}(1+t^3)^{2g-1}}{(1-t)^3(1+t^2)} \\
 & + \frac{2^{2g-1}t^{10g-8}(1+t)^{2g}}{(1-t)^3(1-t^3)} + \frac{t^{4g-4}(1-t)^{2g-1}(1+t)^{2g-1}}{4(1+t^2)} \\
 & + \frac{t^{6g-2}(1+t)^{4g-3}(1+t^2+t^4)}{(t-1)^3(1+t^2)^2(t^6-1)} + \frac{(1+t^5)^{2g}(1+t^3)^{2g-1}}{(t^2-1)(t^4-1)^2(t^3-1)}
 \end{aligned}$$