

Cohomology of Higgs moduli spaces

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Lecture 1: Cohomology of Higgs moduli via \mathbb{C}^\times action

Lecture 2: Cohomology of character variety via arithmetic

Lecture 3: Hard Lefschetz via $P = W$

Ubiquity of Higgs bundles

- [Hitchin 1987] introduces Higgs bundles (E, ϕ) on Riemann surfaces as solutions of a two-dimensional reduction of the 4D Yang–Mills equations \leadsto Hitchin integrable system
- [Simpson 1990] studies Higgs bundles in the framework of non-Abelian Hodge theory of a variety
- [Beilinson–Drinfeld \sim 1995] Geometric Langlands via quantization of Hitchin system
- [Donagi–Witten 1996] & [Moore–Nekrasov–Shatashvili 1997] importance of Higgs moduli in supersymmetric gauge theories
- [Kapustin–Witten 2005] S-duality perspective on Geometric Langlands and Higgs moduli spaces
- [Gaiotto–Neitzke–Moore 2008-] hyperkähler metric on Higgs moduli spaces maybe understood from wall-crossing
- [Ngô, 2010] symmetries of Hitchin fibers \leadsto fundamental lemma
- [Chuang–Diaconescu–Pan 2010-] Donaldson–Thomas theory on local curve Calabi–Yau 3-folds \leadsto Higgs moduli spaces

Moduli space of vector bundles

- C smooth complex projective curve of genus $g > 1$
- fix integers $r > 0$ and $d \in \mathbb{Z}$ always assume $(d, r) = 1$
- $\mathcal{N} :=$ moduli space of isomorphism classes of semi-stable rank r degree d vector bundles on C
- constructed using geometric invariant theory (GIT) or gauge theory
- vector bundle E is called *semi-stable* (*stable*) if every proper subbundle F satisfies

$$\mu(F) = \frac{\deg(F)}{\text{rk}(F)} \stackrel{(<)}{\leq} \mu(E) = \frac{\deg(E)}{\text{rk}(E)}$$

- when $(d, r) = 1$ semi-stability \Leftrightarrow stability $\leadsto \mathcal{N}^d$ is a non-singular projective variety
- \mathbb{E}^d a universal vector bundle on $\mathcal{N}^d \times C$ in the sense that $\mathbb{E}^d|_{\{E\} \times C} \cong E$

- $H^*(\mathcal{N}^d) := H^*(\mathcal{N}^d; \mathbb{C})$ well understood
- [Harder–Narasimhan 1975] and [Atiyah–Bott 1981] obtained recursive formulae for $P(\mathcal{N}^d; t) := \sum_{i=0} \dim(H^i(\mathcal{N}^d))t^i$
- $\leadsto P_t(\mathcal{N}^d)$ depends on d
- [Atiyah–Bott 1981] \leadsto the ring $H^*(\mathcal{N}^d)$ is generated by the Künneth components of $c(\mathbb{E}) \in H^*(\mathcal{N}^d) \otimes H^*(C)$
- [Jeffrey–Kirwan 1998] \leadsto all intersection numbers (integrals of products of universal generators) as conjectured by [Witten 1992]
- [Earl–Kirwan 2005] complete set of relations among the universal generators of $H^*(\mathcal{N}^d)$

Moduli space of Higgs bundles

- (E, ϕ) a Higgs bundle on C
 - E vector bundle on C
 - $\phi \in H^0(C; \text{End}(E) \otimes K_C)$ Higgs field
- (E, ϕ) is called *semi-stable* (*stable*) if every proper ϕ -invariant subbundle $F \subset E$ satisfies

$$\mu(F) = \frac{\deg(F)}{\text{rk}(F)} \stackrel{(<)}{\leq} \mu(E) = \frac{\deg(E)}{\text{rk}(E)}$$

- $\mathcal{M}^d :=$ moduli space of isomorphism classes of semi-stable rank r degree d Higgs bundles on C
- constructed using geometric invariant theory (GIT) or gauge theory
- $(d, r) = 1 \rightsquigarrow \mathcal{M}^d$ is non-singular, quasi-projective & (\mathbb{E}, Φ) universal Higgs bundle on $\mathcal{M}^d \times C$
- \mathbb{C}^\times acts on \mathcal{M}^d by $(E, \phi) \mapsto (E, \lambda\phi)$
 \mathcal{M}^d is *semi-projective*

Semi-projective varieties

- \mathcal{M} smooth quasi-projective variety *semi-projective*
 - has a \mathbb{C}^\times action
 - $\mathcal{M}^{\mathbb{C}^\times}$ proper
 - $\lim_{\lambda \rightarrow 0} \lambda z$ exists for all $z \in \mathcal{M}$
- examples:
 - \mathcal{M}^d
 - semi-projective toric varieties
 - toric hyperkähler varieties
 - Nakajima quiver varieties
- $\mathcal{M}^{\mathbb{C}^\times} = \bigsqcup_{i \in I} F_i$ connected components
- $U_i := \{z \in \mathcal{M} \mid \lim_{\lambda \rightarrow 0} \lambda z \in F_i\}$ affine bundle over F_i
- $\mathcal{M} = \bigsqcup_{i \in I} U_i$ *Bialinicki-Birula decomposition*
- $D_i := \{z \in \mathcal{M} \mid \lim_{\lambda \rightarrow \infty} \lambda z \in F_i\}$ affine bundle over F_i
- $C := \bigsqcup_{i \in I} D_i \subset \mathcal{M}$ proper subvariety: *core* of \mathcal{M}

Cohomology of semi-projective varieties

- $H^*(\mathcal{M}) \cong \bigoplus_{i \in I} H^{*-2\text{codim} F_i}(F_i)$ BB decomposition *perfect*
- $H_{\mathbb{C}^\times}^*(\mathcal{M}) \cong H^*(\mathcal{M}) \otimes H_{\mathbb{C}^\times}^*(pt)$ *equivariantly formal*
- $r : H_{\mathbb{C}^\times}^*(\mathcal{M}) \hookrightarrow H_{\mathbb{C}^\times}^*(\mathcal{M}^{\mathbb{C}^\times})$ *Kirwan injectivity*
- $\int_{\mathcal{M}} \alpha\beta := \sum_i \int_{F_i} \frac{i_{F_i}^*(\alpha\beta)}{E_{\mathbb{C}^\times}(N_{F_i})} \in H_{\mathbb{C}^\times}^*(\mathcal{M}) \otimes_{\mathbb{C}[u]} \mathbb{C}(u)$
is a non-degenerate pairing on $H_{\mathbb{C}^\times}^*(\mathcal{M})$
- $Z := \mathcal{M} //_{\zeta_\infty} \mathbb{C}^\times$ orbifold
 $\overline{\mathcal{M}} := \mathcal{M} \times \mathbb{C} //_{\zeta_\infty} \mathbb{C}^\times = \mathcal{M} \amalg Z$ orbifold compactification
- $H^*(\overline{\mathcal{M}}) \twoheadrightarrow H^*(\mathcal{M})$ surjective
- $C \subset \mathcal{M}$ is deformation retract
- $\rightsquigarrow H^*(\mathcal{M})$ has pure weight filtration

- $[(E, \phi)] \in (\mathcal{M}^d)^{\mathbb{C}^\times}$ is fixed by \mathbb{C}^\times -action $\Leftrightarrow (E, \phi) \cong (E, \lambda\phi)$
- $\leadsto E = E_1 \oplus \cdots \oplus E_k$

and lower-triangular: $\phi|_{E_i} \subset E_{i+1} \otimes K_C$ and $\phi|_{E_k} = 0$

- its *type*: $(\text{rk}E_1, \dots, \text{rk}E_k)$ ordered partition of r

its *multi-degree* : $(\text{deg} E_1, \dots, \text{deg} E_k)$ adding to d

- their locus is denoted $F_{r_1, \dots, r_k}^{d_1, \dots, d_r} \subset (\mathcal{M}^d)^{\mathbb{C}^\times}$

- fixed points of type $(r) \rightsquigarrow \phi = 0$ and E is stable $F_r^d \cong \mathcal{N}^d$
- fixed points of type $(1, 1, \dots, 1)$ are determined by E_1 and the divisors of zeros of $\phi_i \in H^0(E_i^{-1} \otimes E_{i+1} \otimes K_C)$

$$\rightsquigarrow F_{1, \dots, 1}^{d_1, \dots, d_r} \cong J^d(C) \times S^{d_2+2g-2-d_1}(C) \times \dots \times S^{d_r+2g-2-d_{r-1}}(C)$$

- when $r = 2$, only two types:

$$F_2^d \cong \mathcal{N}^d \text{ and}$$

$$F_{1,1}^{d_1, d_2} \cong J^{d_1}(C) \times S^{2d_2+2g-2-d_1}(C) \text{ with stability } \rightsquigarrow d_2 < d/2$$

- Betti numbers ($b_i = \dim H^i(\mathcal{M}^d)$) computed for
 - $r = 2$ [Hitchin 1987] complicated formula (2 lines)
 - $r = 3$ [Göthen 1994] complicated formula (9 lines)
 - $r = 4$ [Garcia-Prada–Heinloth–Schmitt 2011] (7 lines & 20 summation signs)
 - $r > 4$ [Garcia-Prada–Heinloth 2012] claim computation converges
- universal generators for cohomology ring were proved by
 - $r = 2$ [Hausel–Thaddeus 2002]
 - $r > 2$ [Markman 2002]
- relations in cohomology ring described for $r = 2$ by [Hausel–Thaddeus 2002]
- equivariant intersection numbers conjectured by [Moore–Nekrasov–Shatashvili 1997] & [Hausel–Szenes 2008 \leq]