Betti numbers of large hyperkähler manifolds

joint work with Fernando R. Villegas

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How geometry effects topology?

- conjecture of (Hopf, 1930') $\rightsquigarrow$
  compact Riemannian M^{2m} such that $sec(M) > 0 \Rightarrow \chi(M) > 0$

- M^{2m} compact Kähler manifold $\overset{\text{Hodge}}{\Rightarrow}$ odd Betti numbers even
  (e.g. Hopf surface $\cong S^3 \times S^1$ has no Kähler metric)

- M^{2m} Kähler $\Leftrightarrow$ holonomy $\subset U(m) \rightsquigarrow \omega \in \Omega^2(M)$

- Hard Lefschetz (HLT): $$\begin{array}{cccc} L^l: & H^{m-l}(M) & \cong & H^{m+l}(M) \\ & x & \mapsto & x \cup [\omega]^l \end{array}$$

- smooth projective toric variety M^{2m} Kähler
  HLT gives all constraints on Betti numbers (Stanley 1980) $\rightsquigarrow$
  classification of face vectors of simple rational convex polytopes

Hyperkähler manifolds

- M^{4m} hyperkähler when holonomomy $\subset \mathrm{Sp(m)} \Leftrightarrow$
  Kähler wrt complex structures I, J, K satisfying $IJK = -1$
- there are few known compact examples;
  many non-compact but complete examples:
- toric hyperkähler manifolds $\mathcal{M}_{\mathcal{H}}$; e.g. toric quiver varieties $\mathcal{M}_{\mathbf{1}}^{\Gamma}$
- Hilbert scheme of n-points on $\mathbb{C}^2$: $(\mathbb{C}^2)^{[n]}$
- moduli spaces of YM instantons on $\mathbb{R}^4 \rightsquigarrow$ ADHM spaces $\mathcal{M}_{n,m}$
- Nakajima quiver varieties $\mathcal{M}_{\mathbf{v}}^{\Gamma}$
- moduli space of Higgs bundles on a Riemann surface $\mathcal{M}_n^g$
- all of these examples are *semiprojective*

$\Leftrightarrow \mathbb{C}^\times \circlearrowright M_I$ s.t. $\lim_{\lambda \to 0} \lambda z$ exists for all $z \in M$
core $\mathcal{C} := \{x \in M | \lim_{\lambda \to \infty} \lambda z \text{ exists}\}$ projective

Betti numbers of semiprojective hyperkähler manifolds

- (Elingsrud–Strømme 1987) $\rightsquigarrow$
  $b_{2i}((\mathbb{C}^2)^{[n]}) = \#\{\text{partitions of } n \text{ into } i \text{ parts}\}$
- (Bielawski–Dancer 2000) $\rightsquigarrow$
  $b_{2i}(\mathcal{M}_{\mathcal{H}}) = h_i(\mathcal{H})$ h-number of hyperplane arrangement
- (Hausel–Sturmfels 2002) $\rightsquigarrow$
  $P_t(\mathcal{M}^\Gamma_{\mathbf{1}}) = \sum b_{2i}(\mathcal{M}^\Gamma_{\mathbf{1}})t^{2i} = \frac{t^{2d}}{(1-t^2)^{n-1}} Rel_\Gamma(1/t^2)$ *reliability polynomial* of Γ
- (Crawley-Boevey–Van den Bergh 2004) $\rightsquigarrow$
  $P_t(\mathcal{M}^\Gamma_{\mathbf{v}}) = t^{2d_{\mathbf{v}}} A^\Gamma_{\mathbf{v}}(1/t^2)$ *Kac polynomial* counting absolutely indecomposable representations of quiver Γ of dimension **v**
- (Hausel–Villegas 2008) $\rightsquigarrow$ conjecture for $P_t(\mathcal{M}^g_n)$
  (Chuan–Diaconescu–Pan 2012) $\rightsquigarrow$
  our conjecture ⇔ refined Gopakumar-Vafa conjecture
- ⇔ arithmetic harmonic analysis on finite Lie groups, algebras
- all the formulas are combinatorially tractable

Restrictions on Betti numbers of hyperkähler manifolds

- semiprojective hyperkähler manifold $M^{4m} \sim \mathcal{C}$
  $\rightsquigarrow b_i(M^{4m}) = 0$ when $i > \dim \mathcal{C} = 2m$
- (Hausel 2006) $\rightsquigarrow$ semiprojective hyperkähler manifold satisfies *weak Hard Lefschetz*:

$$\begin{array}{cccc} L: & H^i(M^{4m})) & \hookrightarrow & H^{i+2}(M^{4m}) \\ & x & \mapsto & x \cup [\omega_I] \end{array} \quad \text{for } i \leq \min \dim(\mathcal{C})/2 = m$$

- (Hausel-Sturmfels 2002)
  $\rightsquigarrow$ restrictions on Betti numbers of toric hyperkähler varieties
  $\rightsquigarrow$ new restrictions on face vectors of matroid complexes
- weak Hard Lefschetz for $\mathcal{M}_n^g$ also follows from $P = W$ conjecture of (de Cataldo–Hausel–Migliorini 2010)
- What do Poincaré polynomials of semi-projective hyperkähler varieties look like?

Gaussian distribution as limit

- Gaussian distribution: $\exp(-\frac{x^2}{2})$
  e.g. distribution of chest circumference in scottish army 1846
- central limit theorem (CLT) $\rightsquigarrow$ appropriately rescaled, the graphs of the polynomials $P_t(\mathcal{M}_1^g) = (1+t)^{2g}$ tend to the Gaussian distribution
- $P_t(T^*\mathrm{Gr}(n,k)) = \begin{bmatrix} n \\ k \end{bmatrix}_{t^2} = \prod_{i=1}^{k} \frac{1-q^{n+1-i}}{1-q^i}$

  $P_t(T^*\mathrm{Gr}(n,1)) = \begin{bmatrix} n \\ 1 \end{bmatrix}_{t^2} \to \mathbf{1}$ as $n \to \infty$;

  $\mathrm{P}_t(T^*\mathrm{Gr}(n,k)) = \begin{bmatrix} n \\ k \end{bmatrix}_{t^2} \to \mathbf{1}^{*k}$ as $n \to \infty$;

  CLT $\rightsquigarrow \mathbf{1}^{*k} \to$ Gaussian as $k \to \infty$
- conjecture: " $\lim_{m\to\infty} \lim_{n\to\infty}$ " $P_t(\mathcal{M}_{n,m})$ is Gaussian
- (Morrison 2013) $\rightsquigarrow$ appropriately rescaled
  " P_t " $((\mathbb{C}^3)^{[n]}) = DT_n(\mathbb{C}^3; q)$ tend again to Gaussian as $n \to \infty$

Gumbel distribution as limit

- Gumbel distribution: $e^x e^{-e^x}$
  e.g. distribution of maximum annual value of daily rainfalls
- (Ellingsrud–Strømme 1987) $\rightsquigarrow$
  $b_{2i}((\mathbb{C}^2)^{[n]}) = \#\{\text{partitions of } n \text{ into } i \text{ parts}\}$
- (Erdős–Lehner, 1941) $\rightsquigarrow$ the distribution of $\#\{\text{partitions of } n \text{ into } i \text{ parts}\}$ is governed by the Gumbel distribution
- i.e. $P_t((\mathbb{C}^2)^{[n]})$ tends to Gumbel as $n \to \infty$
- $\#\{\text{partitions of } n \text{ into } i \text{ parts}\} =$
  $$= \#\{\text{partitions of } n \text{ with largest part } i\}$$

Airy distribution as limit

- Airy distribution: implicitly given by its (complicated) moments e.g. area under Brownian motion
- (Hausel–Sturmfels, 2002) $\rightsquigarrow P_t(\mathcal{M}_{\mathbf{1}}^{\Gamma}) = t^{2d} A_{\mathbf{1}}^{\Gamma}(1/t^2)$
- when $\Gamma = K_n$ the complete graph on n vertices

$$A_{\mathbf{1}}^{\Gamma}(1+q) = \sum_k c_{n,k} q^k,$$

$c_{n,k} = \#\{\text{connected graphs on } n \text{ labelled vertices of genus } k\}$ are the moments of the discrete distribution given by $A_{\mathbf{1}}^{\Gamma}$
- asymptotics determined by (Wright 1977) $\rightsquigarrow$ $\lim_{n\to\infty} P_t(\mathcal{M}_{\mathbf{1}}^{K_n})$ approaches the Airy distribution
- (Reineke, 2005) computed $P_t(H_{n,1}^{(g)})$ for certain non-commutative Hilbert schemes $H_{n,1}^{(g)}$ limiting behaviour of $P_t(H_{n,1}^{(g)})$ is Airy as $n \to \infty$
- expect $P_t(\mathcal{M}_{\mathbf{v}}^{\Gamma}) \to$ Airy as $\mathbf{v} \to \infty$ in many/most directions

Questions

- Is there a limiting geometrical object $\lim_{n\to\infty} \mathcal{M}_n$ in our examples?
- Does it have a "continuously graded" cohomology, with Poincaré distribution given by the observed limiting distribution?
- Is A_1^{Γ} convergent for every large graph limit sequence Γ_n of Lovász –Szegedy?
- large N 't Hooft limit is $N \to \infty$ in certain $U(N)$ gauge theories, where string theory appears.
  Is any of our limiting observations relevant to this limit?
- $\lim_{n\to\infty} P_t(\mathcal{M}_n^g) = ?$
- $\lim_{n\to\infty} H_{t,q}(\mathcal{M}_n^g) = ?$