Betti numbers of hyperkähler manifolds via arithmetic harmonic analysis

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Motivation: mirror symmetry

- A pair of n dimensional Calabi-Yau manifolds (X, Y) satisfy the topological mirror test if

$$H^{p,q}(X) = H^{n-p,q}(Y)$$

- A pair of n dimensional Calabi-Yau manifolds (X, Y) are Strominger-Yau-Zaslow mirror pairs if they map to the same real n-dimensional manifold B, so that the generic fibers are dual special Lagrangian tori

Diffeomorphic spaces in the non-Abelian Hodge theory of a genus g curve C:

$$\mathcal{M}^d_{\mathsf{Dol}}(GL(n,\mathbb{C})) := \left\{ \begin{array}{c} \text{moduli space of semistable rank } n \\ \text{degree } d \text{ Higgs bundles on } C \end{array} \right\}$$

$$\mathcal{M}^d_{\mathsf{DR}}(GL(n,\mathbb{C})) := \left\{ \begin{array}{c} \text{moduli space of flat } GL(n,\mathbb{C})\text{-connections} \\ \text{on } C \setminus \{p\}, \text{ with holonomy } e^{\frac{2\pi i d}{n}} Id \text{ around } p \end{array} \right\}$$

$$\mathcal{M}^d_{\mathsf{B}}(GL(n,\mathbb{C})) := \{A_1, B_1, \ldots, A_g, B_g \in GL(n,\mathbb{C}) | \\ A_1^{-1}B_1^{-1}A_1B_1 \ldots A_g^{-1}B_g^{-1}A_gB_g = \xi_n Id\}/GL(n,\mathbb{C})$$

Theorem 1 (Hausel–Thaddeus 2003). *In the following diagram*

$$\begin{array}{ccc} \mathcal{M}^d_{\mathsf{Dol}}(PGL(n)) & \longrightarrow & \mathcal{M}^d_{\mathsf{Dol}}(SL(n)) \\ \downarrow \chi_{PGL(n)} & & \downarrow \chi_{SL(n)} \\ \mathcal{H}_{PGL(n)} & \cong & \mathcal{H}_{SL(n)}. \end{array}$$

the generic fibers of the Hitchin maps $\chi_{PGL(n)}$ *and* $\chi_{SL(n)}$ *are dual Abelian varieties.*

$$\Downarrow$$

$\mathcal{M}^d_{\mathsf{DR}}(PGL(n))$ *and* $\mathcal{M}^d_{\mathsf{DR}}(SL(n))$ *satisfy the SYZ construction for a pair of mirror symmetric Calabi-Yau manifolds.*

Topological Mirror Test

Conjecture 1 (Hausel–Thaddeus 2003). *For all* $d, e \in \mathbb{Z}$*, satisfying* $(d,n) = (e,n) = 1$*, we have*

$$E_{\mathsf{st}}^{B^e}(x,y;\mathcal{M}_{\mathsf{DR}}^d(SL(n,\mathbb{C}))) = E_{\mathsf{st}}^{\hat{B}^d}(x,y;\mathcal{M}_{\mathsf{DR}}^e(PGL(n,\mathbb{C}))).$$

Conjecture 2 (Hausel–R-Villegas 2004).

$$E_{\mathsf{st}}^{B^e}(x,y,\mathcal{M}_{\mathsf{B}}^d(SL(n,\mathbb{C}))) = E_{\mathsf{st}}^{\hat{B}^d}(x,y,\mathcal{M}_{\mathsf{B}}^e(PGL(n,\mathbb{C}))).$$

Holomorphic symplectic quotients

- $\mathbb{V}$ finite dimensional complex vector space, G complex reductive group

- representation $\rho : G \to GL(\mathbb{V})$ of G on $\mathbb{V}$, inducing $\varrho : \mathfrak{g} \to \mathfrak{gl}(\mathbb{V})$.

- inducing an action of G on $\mathbb{V} \times \mathbb{V}^*$, preserving the natural symplectic structure, with moment map: $\mu : \mathbb{V} \times \mathbb{V}^* \to \mathfrak{g}^*$ defined by $\mu(v, w)(X) = \langle \varrho(X)v, w \rangle$

- for $\xi \in (\mathfrak{g}^*)^G$ the holomorphic symplectic quotient: $\mathbb{V} \times \mathbb{V}^* ////_\xi G := (\mu^{-1}(\xi))//G$ carries a natural hyperkähler metric at its smooth points

- Examples: affine toric hyperkähler varieties when G abelian; Nakajima's quiver varieties, when ρ is constructed from a quiver, e.g. semisimple adjoint orbits in $\mathfrak{gl}(n, \mathbb{C})$

Group-valued symplectic quotients

$G = GL(n)$; C is a genus g curve with generic semisimple conjugacy classes $\tilde{\mathcal{C}}_1, \ldots, \tilde{\mathcal{C}}_k \subset GL(n, \mathbb{C})$ at punctures $p_1, \ldots, p_k \in C$. The *character variety* is

$$\mathcal{M}_{\mathsf{B}} = \{(A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_k)\, C_i \in \tilde{\mathcal{C}}_i | \\ A_1^{-1} B_1^{-1} A_1 B_1 \ldots A_g^{-1} B_g^{-1} A_g B_g C_1 \ldots C_k = Id\} // G(\mathbb{C})$$

$$\overset{\text{Riemann-Hilbert monodromy map}}{\cong}$$

$$\mathcal{M}_{\mathsf{DR}} := \left\{ \begin{array}{c} \text{moduli space of flat} \\ GL(n, \mathbb{C})\text{-connections on } C \setminus \{p_1, \ldots, p_k\} \\ \text{with holonomy around } p_i \text{ lying in } \tilde{\mathcal{C}}_i \end{array} \right\}$$

Mixed Hodge Structure of Deligne

- $\bigoplus_{p,q} H^{p,q;k}(M)$ is the associated graded to the weight and Hodge filtrations on the cohomology $H^k(M, \mathbb{C})$ of a complex algebraic variety M

- $h^{p,q;k} = \dim(H^{p,q;k}(M))$, the *mixed Hodge numbers*

- $H(M; x, y, t) = \sum_{p,q,k} h^{p,q;k}(M) x^p y^q t^k$, the *mixed Hodge polynomial*

- $P(M; t) = H(M; 1, 1, t)$, the *Poincaré polynomial*

- $E(M; x, y) = x^n y^n H(1/x, 1/y, -1)$, the *E-polynomial* of a smooth variety M.

Arithmetic and topological content of the E-polynomial

Theorem 2 (...,Ito 2004, Katz 2005). *If M is a smooth quasi-projective variety defined over $\mathbb{Z}$ and*

$$\#\{M(\mathbb{F}_q)\} = E(q)$$

is a polynomial in q, then

$$E(M; x, y) = E(xy).$$

- MHS on $H^*(M, \mathbb{C})$ is *pure* if $h^{p,q;k} = 0$ unless $p + q = k \Leftrightarrow H(M; x, y, t) = (xyt^2)^n E(\frac{-1}{xt}, \frac{-1}{yt})$ $\Rightarrow P(M; t) = H(M; 1, 1, t) = t^{2n} E(\frac{-1}{t}, \frac{-1}{t})$; examples of varieties with pure MHS: smooth projective varieties, $\mathcal{M}_{\text{Dol}}$, $\mathcal{M}_{\text{DR}}$, Nakajima's quiver varieties

- in general the pure part of $H(M;x,y,t)$ is
$$PH(M;x,y) = \underset{T^0}{\text{Coeff}}\left(H(M;xT,yT,tT^{-1})\right);$$
which, for a smooth M, is always the image of the cohomology of a smooth compactification

- $M = GL(n,\mathbb{C})$ does not have pure mixed Hodge structure. Indeed $GL(n,\mathbb{C}) \subset \mathfrak{gl}(n,\mathbb{C}) \subset \mathbb{P}^{n^2}$. The cohomology
$$H^*(GL(n,\mathbb{C})) \cong H^*(S^1 \times S^3 \times \cdots \times S^{2n-1})$$
is generated in odd degrees, while $\mathbb{P}^{n^2}$ has only even degree cohomology $\Rightarrow PH(GL(n,\mathbb{C})) = 1$, which is the pure part of
$$H(M;x,y) = (1+xyt)(1+x^2y^2t^3)\cdots(1+x^ny^nt^{2n-1}),$$
consequently
$$E(M;q) = (q^n - q^{n-1})(q^n - q^{n-2})\cdots(q^n - 1)$$
$$P(M;t) = (1+t)(1+t^3)\cdots(1+t^{2n-1})$$

Fourier Transform on finite groups

- Γ finite group; Fourier transform of function $f : G \to \mathbb{C}$ at $\rho : G \to End(V_\rho)$ irrep.:

$$\mathcal{F}(f,\rho) = |\Gamma|^{-1/2} \sum_{g\in\Gamma} f(g)\rho(g) \in End(V_\rho),$$

- the convolution of the functions $f_1, \ldots, f_k : \Gamma \to \mathbb{C}$ is defined as

$$f_1 \star f_2 \star \cdots \star f_k(h) =$$

$$|\Gamma|^{(1-k)/2} \sum_{\substack{g_1,g_2,\ldots,g_k\in\Gamma \\ g_1 g_2 \ldots g_k = h}} f_1(g_1) f_2(g_2) \ldots f_k(g_k)$$

- $\mathcal{F}(f_1 \star f_2, \rho) = \mathcal{F}(f_1,\rho) \circ \mathcal{F}(f_2,\rho)$

- Fourier inversion formula:

$$|\Gamma|^{-1/2} \sum_{\rho\in Irr(\Gamma)} \dim(V_\rho) tr\left(\mathcal{F}(f,\rho) \circ \rho(h^{-1})\right) = f(h)$$

Fourier Transform for $T^*\mathbb{C}P^n$

- Calabi's hyperkähler manifold: $T^*\mathbb{C}P^n \cong \{(v,w) \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} | \sum_{i=1}^{n+1} v_i w_i = 1\}//GL(1)$

- $f(\xi) = q\delta_0 + (q-1)1 =$

  $$\#\{(v,w) \in \mathbb{F}_q \times \mathbb{F}_q | vw = \xi\} = \begin{cases} 2q-1 & \text{if } \xi = 0 \\ q-1 & \text{if } \xi \neq 0 \end{cases}$$

- $\Psi : \mathbb{F}_q \to \mathbb{C}^\times$ additive character; $f : \mathbb{F}_q \to \mathbb{C}$; $Y \in \mathbb{F}_q^*$

  $$\mathcal{F}(f)(Y) = \sum_{X \in \mathbb{F}_q} \Psi(\langle X, Y \rangle)$$

- $\frac{1}{q-1}\#\{(v,w) \in \mathbb{F}_q^{n+1} \times \mathbb{F}_q^{n+1} | \sum_i^{n+1} v_i w_i = 1\} =$

$$\frac{1}{q-1} f \star f \star \cdots \star f(1) = \frac{q^{n/2}}{q-1} \sum_{X \in \mathbb{F}_q} \mathcal{F}(f)(X)^{n+1} \Psi(X)$$

$$= \frac{q^{n/2}}{q-1} \sum_{X \in \mathbb{F}_q} \left(q q^{-1/2} 1(X) + (q-1) q^{1/2} \delta_0(X) \right)^{n+1} \Psi(X)$$

$$= \frac{q^{2n+1} - q^n}{q-1} = q^n (q^n + q^{n-1} + \cdots + 1)$$

- $\Rightarrow P(T^* \mathbb{C}P^n; t) = 1 + t^2 + t^4 + \cdots + t^{2n}$

Fourier Transform for $\mathcal{M}_{\mathsf{B}}$

Setup:

- $G = GL(n)$
- $C = \mathbb{P}^1$, with punctures $a_1, \ldots, a_k \in \mathbb{P}^1$
- $\tilde{\mathcal{C}}_i \subset GL(n)$ fixed semisimple conjugacy classes
- $\mathcal{M}_{\mathsf{B}} =$
  $\{(A_1, A_2, \ldots, A_k)\, A_i \in \tilde{\mathcal{C}}_i | A_1 \cdot \cdots \cdot A_k = 1\} // G(\mathbb{C})$

Theorem 3 (Frobenius 1896, Hausel–Villegas 2004).

$$\#\{\mathcal{M}_{\mathsf{B}}(\mathbb{F}_q)\} = \frac{|Z(G(\mathbb{F}_q))|}{|G(\mathbb{F}_q)|} 1_{\tilde{C}_1} \star \cdots \star 1_{\tilde{C}_k}(1) =$$

$$\sum_{\chi \in Irr(G(\mathbb{F}_q))} \frac{\chi(1)^2 |Z(G(\mathbb{F}_q))|}{|G(\mathbb{F}_q)|^2} \prod_i \frac{\chi(\tilde{\mathcal{C}}_i(\mathbb{F}_q))}{\chi(1)} |\tilde{\mathcal{C}}_i(\mathbb{F}_q)|$$

is a polynomial $E(q)$ in $q \Rightarrow$

$$E(\mathcal{M}_{\mathsf{B}}, x, y) = E(xy)$$

.

Example Assume $n = 3$, and all the conjugacy classes $\tilde{C}_i$ are regular semisimple:

$$
\begin{aligned}
&E(\mathcal{M}_{\mathsf{B}}; q) = \\
&\quad \frac{\left((q+1)\left(q^2+q+1\right)\right)^k}{\left(q^3-1\right)^2\left(q^2-1\right)^2} - \frac{\left(3\,q^2\,(q+1)\right)^k}{q^4\left(q^2-1\right)^2(q-1)^2} \\
&\qquad + 1/3\,\frac{\left(6\,q^3\right)^k}{q^6\,(q-1)^4} + \frac{\left(2\,q^2\left(q^2+q+1\right)\right)^k}{q^4\left(q^3-1\right)^2(q-1)^2} \\
&+ \frac{\left(q^3\,(q+1)\left(q^2+q+1\right)\right)^k}{q^6\left(q^3-1\right)^2\left(q^2-1\right)^2} - \frac{\left(3\,q^3\,(q+1)\right)^k}{q^6\left(q^2-1\right)^2(q-1)^2}.
\end{aligned}
$$

Conjecture 3 (Hausel 2004). *When* $n=3$, $\tilde{C}_i$ *are regular semisimple,* $h^{i-j}_{N-j}=h^{i+j}_{N+j}$ *for*

$$H(\mathcal{M}_{\mathsf{B}},q,t)=\sum h^i_j q^j t^i =$$

$$\frac{\left(\left(qt^2+1\right)\left(q^2t^4+qt^2+1\right)\right)^k}{\left(q^3t^6-1\right)\left(q^3t^4-1\right)\left(q^2t^4-1\right)\left(q^2t^2-1\right)}$$

$$-\frac{\left(3\,q^2t^4\left(qt^2+1\right)\right)^k}{q^4t^8\left(q^2t^4-1\right)\left(q^2t^2-1\right)\left(qt^2-1\right)(q-1)}$$

$$+1/3\,\frac{6^k\left(qt^2\right)^{3\,k}}{q^6t^{12}\left(qt^2-1\right)^2(q-1)^2}$$

$$+\frac{\left(q^2t^4\left(2\,q^2t^2+qt^2+q+2\right)\right)^k}{q^4t^8\left(q^3t^4-1\right)\left(q^3t^2-1\right)\left(qt^2-1\right)(q-1)}$$

$$+\frac{\left(q^3t^6(q+1)\left(q^2+q+1\right)\right)^k}{q^6t^{12}\left(q^3t^2-1\right)\left(q^3-1\right)\left(q^2t^2-1\right)\left(q^2-1\right)}$$

$$-\frac{\left(3\,q^3t^6(q+1)\right)^k}{q^6t^{12}\left(q^2t^2-1\right)\left(q^2-1\right)\left(qt^2-1\right)(q-1)},$$

Remark. A recent paper of **Garcia-Prada, Gothen** and **Muñoz** calculates the Poincaré polynomial of $\mathcal{M}_{Dol}$ the moduli space of rank 3 parabolic Higgs bundles of degree 0 on a Riemann surface of genus g, with three generic parabolic weights at the punctures. They find that for small values of k (and $g = 0$):

$$P(\mathcal{M}_{Dol}; t) = H(\mathcal{M}_B, 1, t)$$

Some History

- $\mathcal{N}$ = moduli space of stable bundles of rank n and degree d on a curve
- **[Newstead 1967]** $\Rightarrow P(\mathcal{N}, t)$, when $n = 2$
- **[Harder 1970]** $\Rightarrow \#(\mathcal{N}(\mathbb{F}_q)) \Rightarrow$ Weil conjectures true for $\mathcal{N}$, when $n = 2$
- **[Harder-Narasimhan 1974]** $\Rightarrow \#(\mathcal{N}(\mathbb{F}_q))$ for all n + (**[Deligne 1974]** $\Rightarrow$ the Weil conjectures) $\Rightarrow$ recursive formula for $P(\mathcal{N}, t)$ for all n
- **[Atiyah-Bott 1983]** $\Rightarrow$ recursive formula for $P(\mathcal{N}, t)$ for all n, via Morse theory and mathematical gauge theory
- **[Hitchin 1987]** introduces $\mathcal{M}_{Dol}$ via gauge theory and calculates $P(\mathcal{M}_{Dol}, t)$ when $n = 2$, via Morse theory
- **[Gothen 1994]** calculates $P(\mathcal{M}_{Dol}, t)$, when $n = 3$ via Morse theory
- **[Göttsche 1990]** $\Rightarrow \#(S^{[n]}(\mathbb{F}_q))$ for a projective surface S and gets $P(S^{[n]}, t)$ via the Weil conjectures

Conjecture 4 (Hausel 2005). *If $\mathcal{M}_{\mathsf{B}}$ is the $GL(n,\mathbb{C})$ character variety of $\mathbb{P}^1$ punctured with k generic regular semisimple conjugacy classes:*

$$H(\mathcal{M}_{\mathsf{B}};q,t) = \sum_{\substack{\lambda^1,\lambda^2,\ldots,\lambda^l \\ |\lambda^1|+\cdots+|\lambda^l|=n}} A(\lambda_1,\ldots,\lambda_l),$$

where

$$A(\lambda_1,\ldots,\lambda_l) =$$

$$\frac{(-1)^{r-1}(r)!\left(n!(qt^2)^{\frac{n(n-1)}{2}}\prod_{i=1}^l \frac{1}{|\lambda^i|!}B^{\lambda^i}_{(1^n)}((qt^2),q)\right)^k}{r_1!r_2!\ldots r_s!(qt^2)^{(k+2g-2)\frac{n(n-1)}{2}}\prod_{i=1}^l c_{\lambda^i}(q,t)c'_{\lambda^i}(q,t)}.$$

Here B^λ_ρ is a linear combination of Macdonald's (q,t) Kostka polynomials, which is defined as

$$B_{\lambda\mu} = \sum_{|\rho|=|\lambda|} K_{\rho\lambda}(q,t)K_{\rho\mu}.$$

Conjecture 5 (Hausel–Letellier–Villegas 2005).
$g \geq 0$, $k > 0$, $\mu = \{\mu^1, \ldots, \mu^k\} \in \mathcal{P}(n)^{\{1,\ldots,k\}}$, $X_1 = \{x_{1,1}, x_{1,2}, \ldots\}, \ldots, X_k = \{x_{k,1}, x_{k,2}, \ldots\}$,

$$V_\mu^n := H(\mathcal{M}_B(C_\mu); q, -t) \frac{(qt^2)^{(2-2g-k)\frac{n(n-1)}{2}}(1-qt)^{2g}}{(1-q)(1-qt^2)} (qt^2)^{\sum_i^k n(\bar{\mu}^i)} m_{\mu^1}(X_1) \ldots m_{\mu^k}(X_k),$$

$$Z_n(q,t) = \exp\left(\sum_{d=1}^{\infty} \sum_{\mu} \frac{1}{d} V_\mu^n(q^d, t^d, X_1^d, \ldots, X_k^d)\right),$$

$$\mathcal{H}_\lambda^g(q,t) = \prod_{x \in d(\lambda)} \frac{(qt^2)^{(2-2g)l(x)}(1-q^{h(x)}t^{2l(x)+1})^{2g}}{(1-q^{h(x)}t^{2l(x)+2})(1-q^{h(x)}t^{2l(x)})}.$$

$$H_\lambda(X; q, t) = \sum_{|\rho|=|\lambda|} K_{\rho\lambda}(q,t) s_\rho(X),$$

Macdonald (q,t)-symmetric functions, $K_{\rho\lambda}(q,t)$ (q,t)-Kostka polynomials, $s_\rho(X)$ Schur and $m_\mu(X)$ monomial symmetric functions

$$\boxed{\prod_{n=1}^{\infty} Z_n(q,t) = \sum_{\lambda \in \mathcal{P}} \left(\prod_{i=1}^k H_\lambda(X_i; q, qt^2)\right) \mathcal{H}_\lambda^g(q,t)}$$

The Riemann-Hilbert map

- $\mathfrak{g} = \mathfrak{gl}(n)$
- $C = \mathbb{P}^1$ with punctures $a_1, \ldots, a_k \in \mathbb{P}^n$
- $\mathcal{C}_i$ semisimple adjoint orbit in $\mathfrak{g}(\mathbb{C})$
- $Q =$ $\{(A_1, \ldots, A_k)\, A_i \in \mathcal{C}_i | A_1 + \cdots + A_k = 0\} // G(\mathbb{C})$, Nakajima's star-shaped quiver variety
- Q is smooth when $\mathcal{C}_i$ are generic
- "$Q \subset \mathcal{M}_{\mathsf{DR}}$", a point in Q gives the meromorphic flat $GL(n, \mathbb{C})$-connection $\sum A_i \frac{dz}{z - a_i}$ on the trivial bundle on C.
- $\tilde{\mathcal{C}}_i = \exp(2\pi i \mathcal{C}_i) \subset G(\mathbb{C})$ is the corresponding conjugacy class
- the *Riemann-Hilbert monodromy map*

$$\nu_a : Q \to \mathcal{M}_{\mathsf{B}}$$

is given by sending the flat connection to its holonomy.

The purity conjecture

Conjecture 6. *If* $\mathcal{C}_i$ *are generic, then*

$$\nu_a^* : PH^*(\mathcal{M}_{\mathsf{B}}) \xrightarrow{\cong} H^*(Q)$$

Example

- $n = 3$, $k = 3$, $\mathcal{C}_i$ regular semisimple
- Q is $\cong$ to an E_6 ALE space,
- $\mathcal{M}_{\mathsf{B}} \cong \mathcal{M}_{\mathsf{Dol}}$ an elliptic fibration with singular fibre of type $\widehat{E}_6$.
- $P_t(Q) = 1 + 6t^2$
- $H(\mathcal{M}_{\mathsf{B}}; q, t) = 1 + 6qt^2 + q^2t^2$

$$\Downarrow$$

Conjecture is true in this case

Fourier transform for Q

- $\Psi : \mathbb{F}_q \to \mathbb{C}$ non-trivial additive character
- $f : \mathfrak{g}(\mathbb{F}_q) \to \mathbb{C}$ its Fourier transform $\mathcal{F}(f) : \mathfrak{g}^*(\mathbb{F}_q) \to \mathbb{C}$ at a $Y \in \mathfrak{g}^*(\mathbb{F}_q)$

$$\mathcal{F}(f)(Y) := |\mathfrak{g}(\mathbb{F}_q)|^{-1/2} \sum_{X \in \mathfrak{g}(\mathbb{F}_q)} f(X)\Psi(\langle X, Y\rangle).$$

- $\delta_{\mathcal{C}_i} : \mathfrak{g}(\mathbb{F}_q) \to \mathbb{C}$ characteristic function of $\mathcal{C}_i \subset \mathfrak{g}(\mathbb{F}_q)$

Theorem 4 (Hausel–Villegas 2004).

$$\#\{Q(\mathbb{F}_q)\} = \frac{|Z(G(\mathbb{F}_q))||\mathfrak{g}(\mathbb{F}_q)|^{\frac{k-2}{2}}}{|G(\mathbb{F}_q)|} \sum_{X \in \mathfrak{g}(\mathbb{F}_q)} \mathcal{F}(\delta_{\mathcal{C}_1})(X) \cdot \dots \cdot \mathcal{F}(\delta_{\mathcal{C}_k})(X)$$

Theorem 5 (Hausel–Letellier–Villegas 2005). *Let* $H_\mu(q,t) \stackrel{?}{=} H(\mathcal{M}_B(C_\mu),q,t)$ *the conjectured mixed Hodge polynomial. Then*

$$H_\mu(q,-1) = E(\mathcal{M}_B(C_\mu),q)$$

from the character table of $GL(n,\mathbb{F}_q)$ *and the pure part*

$$PH_\mu(t^2) = P(Q_\mu,t)$$

from the character table of $\mathfrak{gl}(n,\mathbb{F}_q)$.

Example When $n=3$: $\mu_i = (1^n)$ regular semi-simple

$$P(Q;t) = \frac{\left(\left(t^2+1\right)\left(t^4+t^2+1\right)\right)^k}{\left(t^6-1\right)\left(t^4-1\right)} - \frac{\left(3\,t^4\left(t^2+1\right)\right)^k}{t^8\left(t^4-1\right)\left(t^2-1\right)} + 1/3\,\frac{6^k\left(t^2\right)^{3\,k}}{t^{12}\left(t^2-1\right)^2} - \frac{\left(t^4\left(t^2+2\right)\right)^k}{t^8\left(t^2-1\right)} + t^{6k-12} + \frac{(3\,t^6)^k}{t^{12}\left(t^2-1\right)}.$$

Purity conjecture for $g > 0$

Setup

- C genus g curve, punctures $a_1, \ldots, a_k \in C$
- $\tilde{\mathcal{C}}_i \subset G$ semisimple conjugacy classes
- $\mathcal{M}_{\mathsf{B}} = \{(A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_k)\, C_i \in \tilde{\mathcal{C}}_i |$
  $A_1^{-1} B_1^{-1} A_1 B_1 \ldots A_g^{-1} B_g^{-1} A_g B_g C_1 \ldots C_k = Id\} // G(\mathbb{C})$
- $\mathcal{C}_i \subset \mathfrak{g}$ semisimple adjoint orbit
- $Q = \{(A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_k)\, C_i \in \tilde{\mathcal{C}}_i |$
  $$A_1 B_1 - B_1 A_1 + \cdots + A_g B_g - B_g A_g$$
  $$+ C_1 + \cdots + C_k = 0\} // G(\mathbb{C})$$

Conjecture 7. *The pure part of the cohomology of $\mathcal{M}_{\mathsf{B}}$ is isomorphic with the cohomology of Q, in particular: $PH(\mathcal{M}_{\mathsf{B}}; q, t) = P(Q; t)$*

Fourier transform for $(\mathbb{C}^2)^{[n]}$

- $g = 1$ and $k = 1$, and $\mathcal{C}_1 \subset GL(n)$ is a smallest non-central semisimple orbit
- $\mathcal{M}_{\mathrm{B}} =$

  $$\{(A_1, B_1, C)C \in \mathcal{C}_1 | A_1^{-1} B_1^{-1} A_1 B_1 C = Id\}//GL(n)$$

- $Q \cong (\mathbb{C}^2)^{[n]}$
- Conjecture $\Rightarrow$ $PH^*(\mathcal{M}_{\mathrm{B}}) \cong H^*((\mathbb{C}^2)^{[n]}, \mathbb{C})$.
- **[Nevins-Stafford 2003]** $\Rightarrow$

  $$\mathcal{M}_{\mathrm{B}} \cong (\mathbb{C}^\times \times \mathbb{C}^\times)^{[n]}$$

Conjecture 8 (Hausel 2005). *In this case:*

$$\sum_{n=1}^{\infty} H(\mathcal{M}_{\mathrm{B}}; q, t) T^n = \prod_{m=1}^{\infty} \frac{(1 + q^m t^{2m-1} T^m)^2}{(1 - q^{m-1} t^{2m-2} T^m)(1 - q^{m+1} t^{2m} T^m)}$$

Fourier transform for $\mathbb{V} \times \mathbb{V}^* ////_\xi G$

- representation $\rho : G \to GL(\mathbb{V})$ of G on $\mathbb{V}$, inducing $\varrho : \mathfrak{g} \to \mathfrak{gl}(\mathbb{V})$.

- inducing an action of G on $\mathbb{V} \times \mathbb{V}^*$, preserving the natural symplectic structure, with moment map: $\mu : \mathbb{V} \times \mathbb{V}^* \to \mathfrak{g}^*$ defined by $\mu(v,w)(X) = \langle \varrho(X)v, w\rangle$

Proposition 6 (Hausel 2005). *Let $\xi \in \mathfrak{g}(\mathbb{F}_q)^*$. The number of solutions of the equation $\mu(x) = \xi$ over the finite field $\mathbb{F}_q$ is given by the formula:*

$$\#\{(v,w) \in \mathbb{V}(\mathbb{F}_q) \times \mathbb{V}(\mathbb{F}_q)^* | \mu(v,w) = \xi\} =$$
$$|\mathfrak{g}(\mathbb{F}_q)|^{-1} |\mathbb{V}(\mathbb{F}_q)| \sum_{X \in \mathfrak{g}(\mathbb{F}_q)} C_\varrho(X) \frac{(q\delta_0(\langle X, \xi\rangle) - 1)}{q-1}$$
$$= |\mathfrak{g}(\mathbb{F}_q)|^{-1/2} |\mathbb{V}| \mathcal{F}(C_\varrho)(\xi),$$

where $C_\varrho(X) = |\ker(\varrho(X))|$.

Theorem 7 (Hausel 2005). $Q = (\mathcal{V}, \mathcal{E})$ *quiver,* $\mathcal{V} = \{1, \ldots, n\}$ *vertices* $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ *edges,* $\mathbf{w} \in \mathbb{N}^{\mathcal{V}}$ *dimension vector,* $\mathcal{M}(\mathbf{v}, \mathbf{w})$ *affine Nakajima quiver variety.*

$$\sum_{\mathbf{v} \in \mathcal{V}^{\mathbb{N}}} P_t(\mathcal{M}(\mathbf{v}, \mathbf{w})) t^{-d(\mathbf{v},\mathbf{w})} T^{\mathbf{v}} =$$

$$= \frac{\displaystyle\sum_{\mathbf{v} \in \mathbb{N}^{\mathcal{V}}} T^{\mathbf{v}} \sum_{\lambda \in \mathcal{P}(\mathbf{v})} \frac{\left(\prod_{(i,j) \in \mathcal{E}} t^{-2n(\lambda^i, \lambda^j)}\right)\left(\prod_{i \in \mathcal{V}} t^{-2n(\lambda^i, (1^{\mathbf{w}_i}))}\right)}{\prod_{i \in \mathcal{V}} \left(t^{-2n(\lambda^i, \lambda^i)} \prod_k \prod_{j=1}^{m_k(\lambda^i)} (1 - t^{2j})\right)}}{\displaystyle\sum_{\mathbf{v} \in \mathbb{N}^{\mathcal{V}}} T^{\mathbf{v}} \sum_{\lambda \in \mathcal{P}(\mathbf{v})} \frac{\prod_{(i,j) \in \mathcal{E}} t^{-2n(\lambda^i, \lambda^j)}}{\prod_{i \in \mathcal{V}} \left(t^{-2n(\lambda^i, \lambda^i)} \prod_k \prod_{j=1}^{m_k(\lambda^i)} (1 - t^{2j})\right)}},$$

where $d(\mathbf{v}, \mathbf{w}) = 2\sum_{(i,j) \in \mathcal{E}} \mathbf{v}_i \mathbf{v}_j + 2\sum_{i \in \mathcal{V}} \mathbf{v}_i(\mathbf{w}_i - \mathbf{v}_i)$ *is the dimension of* $\mathcal{M}(\mathbf{v}, \mathbf{w})$, $T^{\mathbf{v}} = \prod_{i \in \mathcal{V}} T_i^{\mathbf{v}_i}$ *and* $n(\lambda, \mu) = \sum_{i,j} \min(\lambda_i, \mu_j)$

Example $\mathcal{V} = \{1\}$, $\mathcal{E} = \{(1,1)\}$, the twisted ADHM space of $U(k)$Yang-Mills instantons of charge n on $\mathbb{R}^4$ is then $\mathcal{M}(n,k) = \mu^{-1}(Id_V)//G$, where

$$\mu(A,B,I,J) = [A,B] + IJ$$

is the ADHM equation. Theorem 7 then implies the result of Nakajima-Yoshioka 2004:

$$\sum_{n=0}^{\infty} P_t(\mathcal{M}(n,k))T^n = \prod_{i=1}^{\infty} \prod_{b=1}^{k} \frac{1}{(1 - t^{2(k(i-1)+b-1)}T^i)}.$$

Food for thought

- G (compact, finite, non-compact reductive) Lie group

$$\mathcal{M}_{\mathsf{B}}(G) := \{A_1, B_1, \ldots, A_g, B_g \in G | \\ A_1^{-1} B_1^{-1} A_1 B_1 \ldots A_g^{-1} B_g^{-1} A_g B_g = Id_G\}/G$$

- Zeta function of G:

$$\zeta_G(s) = \sum_{\chi \in Irr(G)} \mathrm{dim}(\chi)^{-s}$$

- e.g. $\zeta_{SU(2)}(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function

- Character formula

$$Vol(\mathcal{M}_B) := \frac{Vol(G)^{2g-2}}{Vol(Z(G))} \zeta_G(2g-2)$$

- G simple compact group, e.g. $G = SU(n)$; **[Witten 1992]** $\Rightarrow Vol(\mathcal{M}_B)$ is symplectic volume; **[Alekseev, Meinrenken, Woodward, 2002]** use harmonic analysis on G to get Witten's formula

- G finite group of Lie type, e.g. $G = GL(n, \mathbb{F}_q)$, **[Hausel-Villegas 2003]** $\Rightarrow Vol(\mathcal{M}_B)$ is arithmetic volume, gives the E-polynomial $E(\mathcal{M}_B(\mathbb{C}))$; **[Hausel 2005]** $\Rightarrow$ gets the character formula via harmonic analysis on G

- G non-compact reductive Lie group, e.g $G = SL(n, \mathbb{R})$;
  - What is the right character formula?
  - What volume it calculates?
  - Connection to classical Langlands duality?