Arithmetic of wild character varieties

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Conjecture from physics

Conjecture (Chuang–Diaconescu–Donagi–Pantev 2014)

$$\sum_{n\in\mathbb{Z}} \sum_{\mathbf{m}\in\mathcal{P}^k} \text{“}PT(\tilde{Y}_k^{g,r}, n, \mathbf{m}; y)\text{”}\, q^n \prod_{i=1}^{k} \mathbf{x}_i^{\mathbf{m}_i} = Z(\mathbf{x}; (qy)^{-1/2}, (q/y)^{1/2})$$

- C complex projective curve of genus $g \geq 0$ with $k \geq 1$ punctures $\rightsquigarrow$ $\tilde{C}_k$ orbifold curve with k orbifold points
- $\tilde{Y}_k^{g,r} := tot(\mathcal{O}(rp) \oplus K_{\tilde{C}}(-rp)))$ Calabi-Yau 3-orbifold ($r \geq 0$)
- “$PT(\tilde{Y}_k^{g,r}, n, \mathbf{m}; y)$” Pandharipande-Thomas “refined invariant”
- $Z(\mathbf{x}; z, w) := \sum_{\lambda\in\mathcal{P}^k} \mathcal{H}_\lambda^{g,r}(z, w) \prod_{i=1}^k \tilde{H}_{\lambda^i}(\mathbf{x}_i; z^2, w^2)$
- $\mathcal{H}_\lambda^{g,r}(z, w) := \prod \frac{(z^{2a}w^{2l})^r (z^{2a+1} - w^{2l+1})^{2g}}{(z^{2a+2} - w^{2l})(z^{2a} - w^{2l+2})}$
- $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_k)$, where $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots)$
- $\tilde{H}_{\lambda^i}(\mathbf{x}_i; q, t) \in \mathbb{Q}(q, t)[[x_{i1}, x_{i2}, \ldots]]^{S_\infty} =: \mathbb{Q}(q, t)\Lambda(\mathbf{x}_i)$ Macdonald polynomials

Evidence for Conjecture

- refined Gopakumar-Vafa conjecture for $\tilde{Y}_k^{g,r} \rightsquigarrow$

$$\mathrm{Log}(\text{LHS(Conjecture)}) = \sum_{\mu \in \mathcal{P}^k} \frac{PH(\mathcal{M}_{\mathrm{Dol}}^{\mu,r}, z^{-2}, (wz)^2) w^{-d_\mu}}{(1-z^2)(w^2-1)} \prod_{i=1}^{k} m_{\mu_i}(\mathbf{x}_i)$$

- where $\mathcal{M}_{\mathrm{Dol}}^{\mu,r}$ moduli space of stable parabolic Higgs bundles (E,ϕ) on C where $\phi \in H^*(E, \mathrm{End}(E) \otimes K(\sum p_i + rp))$ with quasi-parabolic structure $\mu \in \mathcal{P}_n^k$
- $PH(\mathcal{M}_{\mathrm{Dol}}^{\mu,r}, q, t) := \sum_{i,j \geq 0} q^i t^k \mathrm{dim}\mathrm{Gr}_i^P H_c^k(\mathcal{M}_{\mathrm{Dol}}^{\mu,r})$
- where P is the perverse filtration on $H_c^*(\mathcal{M}_{\mathrm{Dol}}^{\mu,r})$ induced by the Hitchin map $\chi : \mathcal{M}_{\mathrm{Dol}}^{\mu,r} \to \mathbb{A}^{\mu,r}$

Conjecture from arithmetic

- $r = 0$ non-Abelian Hodge theorem $\rightsquigarrow H_c^*(\mathcal{M}_{\mathrm{Dol}}^\mu) \cong H_c^*(\mathcal{M}_{\mathrm{B}}^\mu) \rightsquigarrow$

Conjecture (de Cataldo–Hausel–Migliorini 2012)

$$P = W$$

- W is Deligne's weight filtration on $H_c^*(\mathcal{M}_{\mathrm{B}}^\mu)$
- $WH(\mathcal{M}_{\mathrm{B}}^\mu, q, t) := \sum_{i,j\geq 0} q^{i/2} t^k \dim \mathrm{Gr}_i^W H_c^k(\mathcal{M}_{\mathrm{B}}^\mu)$
- $P = W$ & Conjecture [CDDP, 2014] & refined GV $\leftrightsquigarrow$

Conjecture (Hausel–Letellier–Villegas 2011)

$$\left\langle \mathbb{H}_k^{g,0}(\mathbf{x}; z, w), h_{\mu^1} \otimes \cdots \otimes h_{\mu^k} \right\rangle = \frac{WH(\mathcal{M}_{\mathrm{B}}^\mu, z^{-2}, (wz)^2) w^{-d_\mu}}{(1 - z^2)(w^2 - 1)}$$

- Master generating function:

$$\mathbb{H}_k^{g,r}(\mathbf{x}; z, w) = \mathrm{Log}\left(\sum_{\lambda \in \mathcal{P}^k} \mathcal{H}_\lambda^{g,r}(z, w) \prod_{i=1}^{k} \tilde{H}_{\lambda^i}(\mathbf{x}_i; z^2, w^2) \right) \in \mathbb{Q}(z, w)\Lambda(\mathbf{x})$$

A puzzle

- perverse filtration makes sense on $H^*(\mathcal{M}_{\mathrm{Dol}}^{\mu,r})$ for $r > 0$

Problem

Is there a character variety $\mathcal{M}_{\mathrm{B}}^{\mu,r}$ such that

$$PH(\mathcal{M}_{\mathrm{Dol}}^{\mu,r}; q, t) = WH(\mathcal{M}_{B}^{\mu,r}; q, t)?$$

- $\rightsquigarrow$ symplectic leaves in $\mathcal{M}_{\mathrm{Dol}}^{\mu,r} \xrightarrow{\text{wild NAH}}$ wild character varieties

Wild character varieties

- follow [Boalch 2014, 2007]
- $\mathcal{A}$ meromorphic connection on C order r_i+1 around k_u punctures:
  $A = d\left(\frac{1}{z_i^r}A_{r_i} + \cdots + \frac{1}{z}A_1\right)$ where $A_j \in \mathfrak{t}_n \subset \mathfrak{gl}_n$ A_{r_i} regular
- $G := \mathrm{GL}_n$, $T \subset G$ maximal torus, $T \subset B_+ \subset G$ Borel, $U_+ \subset B_+$ unipotent radical, $B_- \subset G$ opposite Borel, $U_- \subset B_-$
- local Stokes data as q-Hamiltonian $G \times T$ space:

$$\begin{array}{rcccc} \Phi_i : {}_G\mathcal{A}^i_T := & G \times (U_- \times U_+)^{r_i} \times T & \to & G \times T \\ & (C, S_1, \ldots, S_{2r_i}, t) & \mapsto & (C^{-1} t S_{2r_i} \cdots S_1 C, t^{-1}) \end{array}$$

- $\mathcal{C}_l \subset \mathrm{GL}_n$ ss conjugacy classes at k punctures of type $\mu_l \in \mathcal{P}_n$

$$\begin{array}{cccccc} \Phi : & (G \times G)^g \times & \prod \mathcal{C}_l \times & \prod {}_G\mathcal{A}^j_T & \to & G \times T^{k_u} \\ & (A_i, B_i) & (C_l) & (a_j) & \mapsto & \left(\prod [A_i, B_i] \prod C_l \prod \Phi_j(a_j), \prod t_j^{-1}\right) \end{array}$$

- $\mathcal{M}_{\mathrm{B}}^{\mu,\mathbf{r}} := \Phi^{-1}(\xi)//G \times T^{k_u}$ for $\mathbf{r} := (r_1, \ldots, r_{k_u})$
- $\xi = (\xi_0, \xi_1, \ldots, \xi_m) \in Z(G) \times (T^{reg})^{k_u}$

Theorem (Katz, 2008)

X variety defined over $\mathbb{Z}$ s.t. $E(q) := \#X(\mathbb{F}_q) \in \mathbb{Q}[q] \Rightarrow$

$$WH(X; q, -1) = \sum(-1)^k q^{i/2} \dim Gr_i^W H_c^k(X) = E(q)$$

- assume everything G, T, B, U, Φ, ξ defined over $\mathbb{F}_q$
- [Lusztig, 2010] $\rightsquigarrow \# \left\{a \in {}_G\mathcal{A}_T^j \mid \Phi_j(a) = (g, t)\right\} = \mathrm{Tr}(gT_t T_{w_0}^{r_j})$
- $\mathcal{YH} := \mathbb{C}[U \backslash G / U] \cong \mathbb{C}[N(T)] \cong \mathbb{C}[T \rtimes W]$ Yokonuma-Hecke
  $T_t, T_{w_0} \in \mathrm{End}_G(\mathbb{C}[G/U]) \cong \mathcal{YH}$ Hecke operators
  $w_0 \in W \cong S_{n-1}$ longest element
- $\mathrm{Tr}(gT_{\xi_j} T_{w_0}^{2r_j}) = \sum_{\chi \in Irr(\mathcal{YH})} \chi\left(T_{\xi_j} T_{w_0}^{2r_j}\right) \chi^G(g) = \left(1_{C_{\xi_j}} \star \hat{tw}_{r_j}\right)(g)$
- $\#\mathcal{M}_{\mathrm{B}}^{\mu, \mathbf{r}}(\mathbb{F}_q) = \sum_{\chi \in Irr(GL_n(\mathbb{F}_q))} \frac{|G|^{2g-2}}{\chi(1)^{2g-2}} \prod_{i=1}^{k} \frac{\chi(C_i)|C_i|}{\chi(1)} \prod_{j=1}^{m} q^{r_j f_\chi} \frac{\chi(\xi_j)|C(\xi_j)|}{\chi(1)}$

Mixed Hodge polynomials of wild character varieties

Theorem (Hausel–Mereb–Wong 2013)

$$\mu[k_u] := (\mu, \overbrace{(1^n), \ldots, (1^n)}^{k_u}) \in \mathcal{P}_n^{k+k_u} \text{ and } r := \sum r_i$$

$$WH(\mathcal{M}_{\mathrm{B}}^{\mu,\mathbf{r}}; q, -1) = \left\langle \mathbb{H}^{g,r}_{k+k_u}(\mathbf{x}; q^{-1/2}, -q^{1/2}), h_{\mu^1} \otimes \cdots \otimes h_{\mu^k} \otimes h_{(1^n)}^{\otimes k_u} \right\rangle$$

$$\overset{conj}{=} PH(\mathcal{M}_{\mathrm{Dol}}^{\mu[k_u],r}; q, -1)$$

Conjecture (Hausel–Mereb–Wong 2013)

$$WH(\mathcal{M}_{\mathrm{B}}^{\mu,\mathbf{r}}; q, t) = PH(\mathcal{M}_{\mathrm{Dol}}^{\mu[k_u],r}; q, t)$$

- example: $n = 2$, $g = 0$, $k = 0$, $k_u = 1$, $r = 3$
  Theorem $\rightsquigarrow WH(\mathcal{M}_{\mathrm{B}}^{\mu,\mathbf{r}}; q, -1) = q^2 + q + 1$
  Conjecture $\rightsquigarrow WH(\mathcal{M}_{\mathrm{B}}^{\mu,\mathbf{r}}; q, t) = q^2t^4 + qt^2 + t^2$
  checking with [Boalch, 2014] and [Van der Put–Saito 2009]
- compatibility checks with Boalch's "$2 + 1 =$ *tame*" and associated quiver varieties via purity conjecture

Twisted wild character varieties

- follow [Shende, 2014] and [Witten, 2007]
- *twisted* irregular singularity at $p_i \in C$ of a connection A on C
  $A = d\left(\frac{1}{z^{r_i}} A_{r_i} + \cdots + \frac{1}{z} A_1\right)$ where $A_j \in \mathfrak{gl}_n$ and A_{r_i} nilpotent
- assume A diagonalizable on a finite cover and $A_{r_i} \in \mathcal{N}^{reg}$
- $(n, m) \in \mathbb{Z}^2_{>0}$ and $r = m/n$ Katz invariant and $m = \lceil r \rceil n - l$
- $A_{\lceil r \rceil} \in \mathcal{N}_+^{reg} \subset \mathfrak{b}_+ \subset \mathfrak{gl}_n$ and $A^{st}_{\lceil r \rceil - l} \in \mathcal{N}_-^{surtri} \subset \mathfrak{b}_-$ and $A_i = 0$ ow
- e.g. $m = rn$ the formal type $Q_{n,rn} = \frac{A_{\lceil r \rceil}}{z^{\lceil r \rceil}} + \frac{A^{st}_{\lceil r \rceil}}{z^{\lceil r \rceil}}$ is untwisted
- [Shende, 2014] $\rightsquigarrow$ the local Stokes data with topological monodromy $g \in \mathrm{GL}_n$ has count $Tr(gT^m_{w_c})$ over $\mathbb{F}_q$
- here $T_{w_c} \in \mathcal{H} := \mathbb{C}[\mathrm{B} \backslash \mathrm{G} / \mathrm{B}] \cong \mathrm{End}_{\mathrm{G}}(\mathbb{C}[\mathrm{G}/\mathrm{B}])$ Hecke operator of $w_c = (12..n-1) \in S_{n-1} = W$
- $n = rm$ use $T^n_{w_c} = T^2_{w_0} \rightsquigarrow Tr(gT^{rn}_{w_c}) = Tr(gT^{2r}_{w_0})$ as before
- for $m = \lfloor r \rfloor n + 1 \rightsquigarrow Tr(gT^m_{w_c}) = \left(1_{\mathcal{C}_{g_r}} \star 1_{\mathcal{C}_{g_S}} \star \hat{tw}_{\lfloor r \rfloor}\right)(g)$
- g_r reflection, g_S regular semisimple spectrum full Galois orbit

Refined count of twisted wild character varieties

- $\mu \in \mathcal{P}_n^{k_r}$ and $\mathbf{r} = (r_1, \ldots, r_{k_u}, r'_1, \ldots, r'_{k_t}) \in \mathbb{Z}_{>0}^{k_u} \times (\frac{1}{n} + \mathbb{Z}_{\geq 0})^{k_t}$
- $\mathcal{M}_{\mathrm{B}}^{\mu,\mathbf{r}}$ is the corresponding twisted wild character variety

Theorem (Hausel–Mereb–Wong 2014)

$$\mu[k_u, k_t] := (\mu, \overbrace{(1^n), \ldots, (1^n)}^{k_u}, \overbrace{(1^n), (n-1, 1), \ldots}^{k_t}) \in \mathcal{P}_n^{k_r+k_u+k_t}$$

$$WH(\mathcal{M}_{\mathrm{B}}^{\mu,\mathbf{r}}; q, -1) = \left\langle \mathbb{H}_k^{g,r}(\mathbf{x}; q^{\frac{-1}{2}}, -q^{\frac{1}{2}}), h_{\mu[k_u]} \otimes (h_{(n-1,1)} \otimes p_{(n)})^{\otimes k_t} \right\rangle$$

Conjecture (Hausel–Mereb–Wong 2014)

$$\text{“}WH\text{”}(\mathcal{M}_{\mathrm{B}}^{\mu,\mathbf{r}}; q, t) = tr_{S_{n-1}^{k_t}}(PH(\mathcal{M}_{\mathrm{Dol}}^{\mu[k_u,k_t],r}; q, t))$$

- conjectured “WH”$(\mathcal{M}_{\mathrm{B}}^{\mu,\mathbf{r}}; q, t) \in \mathbb{Z}_{\geq 0}[q, t]$ when $g = 0, k_t = 1$ ow could have negative coefficients
- [Villegas, 2014] $\rightsquigarrow$ for $g = 0, k_r = k_u = 0, k_t = 1$ conjectured “WH”$(\mathcal{M}_{\mathrm{B}}^{\mu,\mathbf{r}}; 1/q, \sqrt{qt})t^{-2d_{\mu,\mathbf{r}}} = C_n^{\lfloor r \rfloor}(q, t)$ (q, t)-Catalan numbers of [Garsia–Haiman, 1996]

Superpolynomials of links

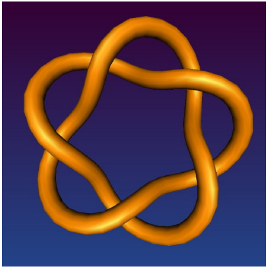
- around singularities of $A = dQ$ [Shende, 2014] associates the link L_Q in $T_1^*(C) :=$ link of the singularity of $\chi_y(Q)$
- e.g. semisimple regular singularity $\rightsquigarrow$ unlink
  model untwisted irregular singularity $\rightsquigarrow$ $T_{n,rn}$ torus link
  model twisted irregular singularity $\rightsquigarrow$ $T_{n,m}$ torus link
  e.g. $(n,m) = 1$ torus knot $T_{n,m}$
- moduli space $\mathcal{M}_{\mathrm{B}}^{Q}$ of Stokes data $\cong$ moduli of rank 1 constructible sheaves on C with singular support at L_Q [Shende–Treumann–Zaslow 2014]
- when $g = 0$, $k = 1$, L_Q knot and $T_{L_Q} = T_{w_0}^2 T_{\tilde{L}_Q} \rightsquigarrow$

$$WH(\mathcal{M}_{\mathrm{B}}^{Q}; 1/z^2, zw)w^{-d_{\mu,\mathrm{r}}} \overset{\text{conj STZ}}{=\!=\!=} (\tfrac{a}{z})^{w-n} P(\tilde{L}_Q; a, z, w)|_{a=0}$$

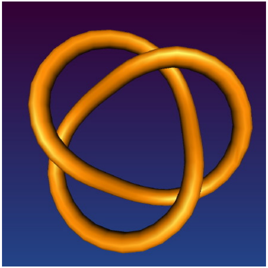
- $P(\tilde{L}_Q; a, z, w)$ superpolynomial knows all refined link invariants
- e.g. $(\frac{a}{z})^{w-n} P(T_{n,nr+1}; a, z, w)|_{a=0} = C_n^r(z^2, w^2)$
  the (q,t)-Catalan numbers

Example

- $g = 0$, $n = 2$, $k_r = k_u = 0$, $\mathbf{r} = (5/2)$



$L_Q = T_{2,5}$



$\tilde{L}_Q = T_{2,3}$

- Theorem $\rightsquigarrow WH(\mathcal{M}_{\mathrm{B}}^{\mathbf{r}}; 1/q, -1)q = q + 1/q = C_2^1(q, 1/q)$
- Conj $\rightsquigarrow WH(\mathcal{M}_{\mathrm{B}}^{\mathbf{r}}; 1/z^2, zw)w^{-2} = z^2 + w^2 = C_2^1(z^2, w^2)$
- checks with [Van der Put–Saito 2009]
- and with $P(T_{2,3}; a, z, w) = \frac{a}{z}\left(z^2 + w^2 + azw^3\right)$