Arithmetic Harmonic Analysis on holomorphic symplectic quotients

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June 2008
Conference on Moment Maps
CRM, Barcelona

General strategy

- Count the number of solutions of moment map equations over finite fields $\mathbb{F}_q \Leftarrow$ use Fourier transform!
- Find that the count is polynomial in q
- Deduce that it agrees with the E-polynomial
- When MHS is pure get Betti numbers of the holomorphic symplectic quotient
  (e.g. for holomorphic symplectic quotients of finite dimensional linear actions)
- When MHS is not pure, get a conjecture for mixed Hodge numbers of the holomorphic symplectic quotient
  (e.g. for group-valued holomorphic quotients)

Mixed Hodge Structure of Deligne

- $\bigoplus_{p,q} H^{p,q;k}(M)$ is the associated graded to the weight and Hodge filtrations on the cohomology $H^k(M, \mathbb{C})$ of a complex algebraic variety M
- $h^{p,q;k} = \dim(H^{p,q;k}(M))$, *mixed Hodge numbers*
- $H(M; x, y, t) = \sum_{p,q,k} h^{p,q;k}(M) x^p y^q t^k$, *mixed Hodge polynomial*
- $P(M; t) = H(M; 1, 1, t)$, *Poincaré polynomial*
- $E(M; x, y) = x^n y^n H(1/x, 1/y, -1)$, *E-polynomial* of a smooth variety M.

Arithmetic and topological content of the E-polynomial

Theorem (Katz 2006)

If M is a smooth quasi-projective variety defined over $\mathbb{Z}$ and

$$\#\{M(\mathbb{F}_q)\} = E(q)$$

is a polynomial in q, then $E(M; x, y) = E(xy)$.

- MHS on $H^*(M, \mathbb{C})$ is *pure* if $h^{p,q;k} = 0$ unless $p + q = k \Leftrightarrow$ $H(M; x, y, t) = (xyt^2)^n E(\frac{-1}{xt}, \frac{-1}{yt}) \Rightarrow$ $P(M; t) = H(M; 1, 1, t) = t^{2n} E(\frac{-1}{t}, \frac{-1}{t})$; examples: smooth projective varieties, $\mathcal{M}_{\mathrm{Dol}}$, $\mathcal{M}_{\mathrm{DR}}$, Nakajima's quiver varieties
- in general the pure part of $H(M; x, y, t)$ is $PH(M; x, y) = \mathrm{Coeff}_{T^0}\left(H(M; xT, yT, tT^{-1})\right)$; which, for a smooth M, is always the image of the cohomology of a smooth compactification

Example

- Let $\mathcal{U}$ be the variety $x_1y_1 + x_2y_2 = 1$ in $\mathbb{C}^2 \times \mathbb{C}^2$.
  $\mathrm{GL}(1)$ acts on $\mathcal{U}$ by
  $\lambda(x_1, x_2, y_1, y_2) \mapsto (\lambda x_1, \lambda x_2, \lambda^{-1}y_1, \lambda^{-1}y_2)$,
  take affine GIT quotient $\mathcal{M} = \mathcal{U}//\mathrm{GL}(1)$
- the number of solutions of the equation $x_1y_1 + x_2y_2 = 1$ in $\mathbb{F}_q$
  is $2(2q-1)(q-1) + (q-2)(q-1)^2 = (q-1)(q^2+q)$
  because
  - $(2q-1)(q-1)$ when $x_1y_1 = 0$
  - $(q-1)(2q-1)$ when $x_1y_1 = 1$
  - $(q-1)^2$ in the other $q-2$ cases.
- $\Rightarrow$ the number of points on $\mathcal{M}(\mathbb{F}_q)$ is q^2+q,
  $\overset{Katz}{\Rightarrow} E(\mathcal{M}, q) = q^2 + q$,
  & MHS is pure on $H^*(\mathcal{M})$,
  $\Rightarrow P(\mathcal{M}, t) = 1 + t^2$
- $\mathcal{M}$ is the holomorphic symplectic quotient associated to the
  Eguchi-Hanson hyperkähler metric $\Rightarrow \mathcal{M} \cong T^*\mathbb{CP}^1$.
  Checks with $P(\mathcal{M}, t) = 1 + t^2$.

Holomorphic symplectic quotients

- $\mathbb{V}$ finite dimensional complex vector space, G complex reductive group
- representation $\rho : G \to GL(\mathbb{V})$ of G on $\mathbb{V}$, with derivative $\varrho : \mathfrak{g} \to \mathfrak{gl}(\mathbb{V})$.
- inducing an action of G on $\mathbb{V} \times \mathbb{V}^*$ preserving the natural symplectic structure with moment map $\mu : \mathbb{V} \times \mathbb{V}^* \to \mathfrak{g}^*$ defined by $\mu(v, w)(X) = \langle \varrho(X)v, w \rangle$
- for $\xi \in (\mathfrak{g}^*)^G$ $\mathbb{V} \times \mathbb{V}^* ////_\xi G = (\mu^{-1}(\xi))//G$ the holomorphic symplectic quotient carries a natural hyperkähler metric at its smooth points
- Examples: affine toric hyperkähler varieties, when G abelian; Nakajima's quiver varieties, when ρ is constructed from a quiver, e.g. semisimple adjoint orbits in $\mathfrak{gl}(n, \mathbb{C})^*$

Fourier transform on $\mathfrak{g}$

- $\Psi : \mathbb{F}_q \to \mathbb{C}$ non-trivial additive character
- $f : \mathfrak{g}(\mathbb{F}_q) \to \mathbb{C}$ its Fourier transform $\hat{f} : \mathfrak{g}^*(\mathbb{F}_q) \to \mathbb{C}$

$$\hat{f}(Y) := \sum_{X \in \mathfrak{g}(\mathbb{F}_q)} f(X)\Psi(\langle X, Y \rangle).$$

- Fourier inversion formula: $\hat{\hat{f}}(h) = |\mathfrak{g}| f(h^{-1})$
- For $\xi \in \mathfrak{g}^*(\mathbb{F}_q)$ the count function of the moment map $\mu : \mathbb{V}(\mathbb{F}_q) \times \mathbb{V}^*(\mathbb{F}_q) \to \mathfrak{g}^*(\mathbb{F}_q)$
  $\#_\mu(\xi) = \#\{(v, w) \in \mathbb{V}(\mathbb{F}_q) \times \mathbb{V}^*(\mathbb{F}_q) | \mu(v, w) = \xi\}$

Proposition (Hausel, 2006)

Recall $\varrho : \mathfrak{g}(\mathbb{F}_q) \to \mathfrak{gl}(\mathbb{V}(\mathbb{F}_q))$ *and let* $a_\varrho = |\ker \varrho|$ *then*

$$\hat{\#}_\mu(x) = |\mathbb{V}| a_\varrho(-x) \quad \overset{\text{Fourier inversion}}{\Longrightarrow} \quad \#_\mu = \frac{|\mathbb{V}|}{|\mathfrak{g}|} \hat{a}_\varrho$$

Examples

- $\varrho : \mathfrak{gl}(1) \to \mathfrak{gl}(\mathbb{C}^2)$ by $(\alpha) \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$
  $a_\varrho : \mathbb{F}_q \to \mathbb{C}$ is $a_\varrho(\alpha) = 1$ unless $\alpha = 0$ when $a_\varrho(0) = q^2$
  $a_\varrho = 1 + (q^2 - 1)\delta_0$ and so $\hat{a}_\mu = q\delta_0 + (q^2 - 1)$.
  Now $\mu : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathfrak{gl}(1)^*$ is given by $x_1 y_1 + x_2 y_2$.
  Recall $\mathcal{U} = \mu^{-1}(1)$. Indeed
  $\#\mathcal{U}(\mathbb{F}_q) = \#_\mu(1) = \frac{q^2}{q} \hat{a}_\rho(1) = q(q^2 - 1) = (q-1)(q^2 + q)$
- $\Gamma = (\mathcal{V}, \mathcal{E})$ quiver $\mathcal{V} = \{1, \ldots, n\}$ vertices $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ edges
  $\mathbf{v}, \mathbf{w} \in \mathbb{N}^{\mathcal{V}}$ and $dim(V_i) = \mathbf{v}_i$ and $\dim(W_i) = \mathbf{w}_i$ then
  $\mathrm{G}_\mathbf{v} = \times_{i \in \mathcal{V}} \mathrm{GL}(V_i)$ naturally acts on
  $\mathbb{V}_{\mathbf{v},\mathbf{w}} = \bigoplus_{(i,j) \in \mathcal{E}} \mathrm{Hom}(V_i, V_j) \oplus \bigoplus_{i \in \mathcal{V}} \mathrm{Hom}(W_i, V_i)$
  the corresponding holomorphic symplectic quotient

  $$\mathcal{M}(\mathbf{v}, \mathbf{w}) = \mu^{-1}(\mathbf{1}_\mathbf{v}) /\!\!/ \mathrm{G}_\mathbf{v}$$

  is the affine *Nakajima quiver variety*

Theorem (Hausel 2006)

$$\sum_{\mathbf{v}\in\mathcal{V}^{\mathbb{N}}} P_t(\mathcal{M}(\mathbf{v},\mathbf{w}))t^{-d(\mathbf{v},\mathbf{w})}T^{\mathbf{v}} =$$

$$= \frac{\displaystyle\sum_{\mathbf{v}\in\mathbb{N}^{\mathcal{V}}} T^{\mathbf{v}} \sum_{\lambda\in\mathcal{P}(\mathbf{v})} \frac{\left(\prod_{(i,j)\in\mathcal{E}} t^{-2\langle\lambda^i,\lambda^j\rangle}\right)\left(\prod_{i\in\mathcal{V}} t^{-2\langle\lambda^i,(1^{\mathbf{w}_i})\rangle}\right)}{\prod_{i\in\mathcal{V}}\left(t^{-2\langle\lambda^i,\lambda^i\rangle)}\prod_k\prod_{j=1}^{m_k(\lambda^i)}(1-t^{2j})\right)}}{\displaystyle\sum_{\mathbf{v}\in\mathbb{N}^{\mathcal{V}}} T^{\mathbf{v}} \sum_{\lambda\in\mathcal{P}(\mathbf{v})} \frac{\prod_{(i,j)\in\mathcal{E}} t^{-2\langle\lambda^i,\lambda^j\rangle}}{\prod_{i\in\mathcal{V}}\left(t^{-2\langle\lambda^i,\lambda^i\rangle}\prod_k\prod_{j=1}^{m_k(\lambda^i)}(1-t^{2j})\right)}},$$

where $d(\mathbf{v},\mathbf{w}) = 2\sum_{(i,j)\in\mathcal{E}}\mathbf{v}_i\mathbf{v}_j + 2\sum_{i\in\mathcal{V}}\mathbf{v}_i(\mathbf{w}_i-\mathbf{v}_i)$ *is the dimension of* $\mathcal{M}(\mathbf{v},\mathbf{w})$, $T^{\mathbf{v}} = \prod_{i\in\mathcal{V}} T_i^{\mathbf{v}_i}$ *and* $\langle\lambda,\mu\rangle = \sum_{i,j}\min(\lambda_i,\mu_j)$

Corollary (Hausel 2006)

The above is a deformation of the Weyl-Kac character formula for highest weight representations of Kac-Moody algebras $\Rightarrow$ *conjecture of [Kac 1982] in representation theory of quivers*

Star-shaped quiver varieties

- $\mathfrak{g} = \mathfrak{gl}(n)$
- $\mathcal{C}_i$ semisimple adjoint orbit in $\mathfrak{g}(\mathbb{C})$
- $Q = \{(A_1, \ldots, A_k)\, A_i \in \mathcal{C}_i | A_1 + \cdots + A_k = 0\} // G(\mathbb{C})$, holomorphic symplectic quotient *star-shaped quiver variety*,
- Q is smooth when $\mathcal{C}_i$ are generic
- $1_{\mathcal{C}_i} : \mathfrak{g}(\mathbb{F}_q) \to \mathbb{C}$ characteristic function of $\mathcal{C}_i \subset \mathfrak{g}(\mathbb{F}_q)$

Theorem

$$|\mathrm{PGL}(\mathrm{n}, \mathbb{F}_\mathrm{q})| \#\{\mathrm{Q}(\mathbb{F}_\mathrm{q})\} = \sum_{\substack{A_1,\ldots,A_k \in \mathfrak{g}(\mathbb{F}_q) \\ A_1 + \cdots + A_k = 0}} 1_{\mathcal{C}_1}(\mathrm{A}_1) \ldots 1_{\mathcal{C}_\mathrm{k}}(\mathrm{A}_\mathrm{k}) = 1_{\mathcal{C}_1} \star \cdots \star 1_{\mathcal{C}_\mathrm{k}}(0) =$$

$$= \frac{1}{|\mathfrak{gl}(n, \mathbb{F}_q)|} \widehat{\hat{1}_{\mathcal{C}_1} \cdots \hat{1}_{\mathcal{C}_k}}(0) = \frac{1}{|\mathfrak{gl}(n, \mathbb{F}_q)|} \sum_{X \in \mathfrak{g}(\mathbb{F}_q)} \hat{1}_{\mathcal{C}_1}(X) \cdot \cdots \cdot \hat{1}_{\mathcal{C}_k}(X)$$

To HAUSEL

Emmanuel Letellier:

Character table of $Gl_3(\mathbb{F}_q)$

	$\begin{pmatrix}a&0&0\\0&a&0\\0&0&a\end{pmatrix}$ $a\in\mathbb{F}_q$	$\begin{pmatrix}a&0&0\\0&a&0\\0&0&b\end{pmatrix}$ $a\neq b$, $a,b\in\mathbb{F}_q$	$\begin{pmatrix}a&0&0\\0&b&0\\0&0&c\end{pmatrix}$ $a,b,c\in\mathbb{F}_q$, $a\neq b, a\neq c, b\neq c$	$\begin{pmatrix}t_1^q&0&0\\0&t_1&0\\0&0&t_2\end{pmatrix}$ $t_1\in\mathbb{F}_{q^2}-\mathbb{F}_q$, $t_2\in\mathbb{F}_q$	$\begin{pmatrix}t_1&0&0\\0&t_1^q&0\\0&0&t_1^{q^2}\end{pmatrix}$ $t_1\in\mathbb{F}_{q^3}-\mathbb{F}_{q^2}$	$\begin{pmatrix}a&0&0\\0&a&1\\0&0&a\end{pmatrix}$ $a\in\mathbb{F}_q$	$\begin{pmatrix}a&1&0\\0&a&1\\0&0&a\end{pmatrix}$ $a\in\mathbb{F}_q$	$\begin{pmatrix}a&0&0\\0&b&1\\0&0&b\end{pmatrix}$ $a\neq b$, $b,a\in\mathbb{F}_q$
$\mathcal{F}\begin{pmatrix}\alpha&0&0\\0&\alpha&0\\0&0&\alpha\end{pmatrix}$ $\alpha\in\mathbb{F}_q$	$q^{-9/2}\psi(a\alpha)^3$	$q^{-9/2}\psi(a\alpha)^2\psi(b\alpha)$	$q^{-9/2}\psi(a\alpha+b\alpha+c\alpha)$	$q^{-9/2}\psi(\alpha(t_1+t_1^q))\psi(\alpha t_2)$	$q^{-9/2}\psi(\alpha(t_1+t_1^q+t_1^{q^2}))$	$q^{-9/2}\psi(a\alpha)^3$	$q^{-9/2}\psi(a\alpha)^3$	$q^{-9/2}\psi(a\alpha)\psi(b\alpha)$
$\mathcal{F}\begin{pmatrix}\alpha&0&0\\0&\alpha&0\\0&0&\beta\end{pmatrix}$ $\alpha\neq\beta$, $\alpha,\beta\in\mathbb{F}_q$	$q^{-5/2}(q^2+q+1)\times\psi(a\alpha)^2\psi(b\beta)$	$q^{-5/2}\psi(\alpha a)^2\psi(\beta b)+(q+1)\psi(\alpha b)\psi(\alpha a)\times\psi(\beta a)$	$q^{-5/2}[\psi(\alpha a+\alpha b+\beta c)+\psi(\alpha a+\alpha c+\beta b)+\psi(\alpha b+\alpha c+\beta a)]$	$q^{-5/2}\psi(\alpha(t_1+t_1^q))\times\psi(\beta t_2)$	0	$q^{-5/2}(1+q)\psi(\alpha a)^2\psi(a\beta)$	$q^{-5/2}\psi(\alpha a)^2\psi(a\beta)$	$q^{-5/2}[\psi(\alpha b)^2\psi(\beta a)+\psi(\alpha b)\psi(\alpha a)\psi(a\beta)$
$\mathcal{F}\begin{pmatrix}\alpha&0&0\\0&\beta&0\\0&0&\gamma\end{pmatrix}$ $\alpha,\beta,\gamma\in\mathbb{F}_q$, $\alpha\neq\beta, \alpha\neq\gamma, \beta\neq\gamma$	$q^{-3/2}(q+1)(q^2+q+1)\psi(\alpha a)\psi(\beta a)\psi(\gamma a)$	$q^{-3/2}(q+1)\times[\psi(\alpha a)\psi(\beta a)\psi(\gamma b)+\psi(\alpha a)\psi(\beta b)\psi(\gamma a)+\psi(\alpha b)\psi(\beta a)\psi(\gamma a)$	$q^{-3/2}\sum_{\sigma\in S_3}\psi(\alpha\sigma(a))\psi(\beta\sigma(b))\psi(\gamma\sigma(c))$	0	0	$q^{-3/2}(1+2q)\psi(\alpha a)\psi(\beta a)\times\psi(\gamma a)$	$q^{-3/2}\psi(\alpha a)\psi(\beta a)\times\psi(\gamma a)$	$q^{-3/2}[\psi(\alpha a)\psi(\beta b)\psi($ $+\psi(\alpha b)\psi(\beta b)\psi($ $+\psi(\alpha b)\psi(\beta a)\psi($
$\mathcal{F}\begin{pmatrix}k_1^q&0&0\\0&k_1&0\\0&0&k_2\end{pmatrix}$ $k_1\in\mathbb{F}_{q^2}-\mathbb{F}_q$, $k_2\in\mathbb{F}_q$	$q^{-3/2}(q^3-1)\psi(a(k_1+k_1^q))\psi(ak_2)$	$q^{-3/2}(q-1)\psi(a(k_1+k_1^q))\psi(bk_2)$	0	$-q^{-3/2}[\psi(k_1^qt_1^q+k_1t_1)\psi(k_2t_2)+\psi(k_2t_2)\psi(k_1t_1^q+k_1^qt_1)]$	0	$-q^{-3/2}\psi(ak_2)\times\psi(a(k_1+k_1^q))$	$-q^{-3/2}\psi(ak_2)\times\psi(a(k_1+k_1^q))$	$-q^{-3/2}\psi(bk_2)\times\psi(a(k_1+k_1^q))$
$\mathcal{F}\begin{pmatrix}k&0&0\\0&k^q&0\\0&0&k^{q^2}\end{pmatrix}$ $k\in\mathbb{F}_{q^3}-\mathbb{F}_{q^2}$	$q^{-3/2}(q^2-1)(q-1)\psi(a(k+k^q+k^{q^2}))$	0	0	0	$\psi(kt_1+(kt_1)^q+(kt_1)^{q^2})+\psi(kt_1^q+k^qt_1^{q^2}+k^{q^2}t_1)+\psi(kt_1^{q^2}+k^qt_1+k^{q^2}t_1^q)$	$q^{-3/2}(1-q)\psi(a(k+k^q+k^{q^2}))$	$q^{-3/2}\psi(a(k+k^q+k^{q^2}))$	0
$\mathcal{F}\begin{pmatrix}\alpha&0&0\\0&\alpha&1\\0&0&\alpha\end{pmatrix}$ $\alpha\in\mathbb{F}_q$	$q^{-3/2}(q^3-1)(q+1)\times\psi(a\alpha)^3$	$q^{-9/2}(q^2-1)(q+1)\psi(a\alpha)^2\psi(\alpha b)$	$q^{-9/2}(2q+1)(q-1)\psi(\alpha a)\psi(\alpha b)\psi(\alpha c)$	$-q^{-9/2}(q+1)\psi(\alpha(t_1+t_1^q))\psi(\alpha t_2)$	$-q^{-9/2}(q^2+q+1)\psi(\alpha(t_1+t_1^q+t_1^{q^2}))$	$q^{-3/2}(q^3-q-1)\psi(\alpha a)^3$	$-q^{-3/2}(q+1)\psi(\alpha a)^3$	$q^{-9/2}(q^2-q-1)\psi(\alpha a)\psi(\alpha b)^2$
$\mathcal{F}\begin{pmatrix}\alpha&1&0\\0&\alpha&1\\0&0&\alpha\end{pmatrix}$ $\alpha\in\mathbb{F}_q$	$q^{-3/2}(q^3-1)(q^2-1)\times\psi(a\alpha)^3$	$q^{-7/2}(q-1)^2(q+1)\psi(\alpha b)\psi(\alpha a)^2$	$q^{-7/2}(q-1)^2\psi(\alpha a)\psi(\alpha b)\psi(\alpha c)$	$-q^{-7/2}(q^2-1)\psi(\alpha(t_1+t_1^q))\psi(\alpha t_2)$	$q^{-7/2}(q^2+q+1)\psi(\alpha(t_1+t_1^q+t_1^{q^2}))$	$q^{-7/2}(1-q^2)\psi(\alpha a)^3$	$q^{-7/2}\psi(\alpha a)^3$	$q^{-7/2}(1-q)\psi(\alpha a)\psi(\alpha b)^2$
$\mathcal{F}\begin{pmatrix}\alpha&0&0\\0&\beta&1\\0&0&\beta\end{pmatrix}$ $\alpha\neq\beta$, $\alpha,\beta\in\mathbb{F}_q$	$q^{-5/2}(q^4+q^3-q-1)\psi(\alpha a)\psi(\beta a)^2$	$q^{-5/2}(q^2-1)\times[\psi(\alpha b)\psi(\beta a)^2+\psi(\alpha a)\psi(\beta a)\psi(\beta b)]$	$q^{-5/2}(q-1)[\psi(\alpha a+\beta b+\beta c)+\psi(\alpha b+\beta c+\beta a)+\psi(\alpha c+\beta a+\beta b)]$	$-q^{-5/2}(q+1)\psi(\beta(t_1+t_1^q))\psi(\alpha t_2)$	0	$q^{-5/2}(q^2-q-1)\psi(\beta a)^2\psi(\alpha a)$	$-q^{-5/2}\psi(\beta a)^2\psi(\alpha a)$	$-q^{-5/2}[\psi(\beta b)^2\psi(\alpha$ $(1-q)\psi(\alpha b)\psi(\beta a)\psi$

Example

Theorem

Letellier's character table for $\mathfrak{gl}(3,\mathbb{F}_q)$ implies that when $n=3$ and all adjoint orbits $\mathcal{C}_i$ are regular semi-simple

$$P(Q;t) = \frac{\left(\left(t^2+1\right)\left(t^4+t^2+1\right)\right)^k}{\left(t^6-1\right)\left(t^4-1\right)} - \frac{\left(3\,t^4\left(t^2+1\right)\right)^k}{t^8\left(t^4-1\right)\left(t^2-1\right)} + 1/3\,\frac{6^k\left(t^2\right)^{3\,k}}{t^{12}\left(t^2-1\right)^2} - \frac{\left(t^4\left(t^2+2\right)\right)^k}{t^8\left(t^2-1\right)} + t^{6k-12} + \frac{\left(3\,t^6\right)^k}{t^{12}\left(t^2-1\right)}$$

e.g. when $k=3$, $P(Q;t)=1+6t^2$, when Q is diffeomorphic to an E_6 ALE space.

Harmonic analysis on finite groups

- Γ finite group; the *convolution* of $f_1, \ldots, f_k : \Gamma \to \mathbb{C}$ is defined:

$$f_1 \star f_2 \star \cdots \star f_k(h) = \sum_{\substack{g_1, g_2, \ldots, g_k \in \Gamma \\ g_1 g_2 \ldots g_k = h}} f_1(g_1) f_2(g_2) \ldots f_k(g_k)$$

- Fourier transform:

$$f : \mathcal{C}(\Gamma) \to \mathbb{C} \rightsquigarrow \hat{f} : \mathrm{Irr}(\Gamma) \to \mathbb{C} \qquad \hat{f} : \mathrm{Irr}(\Gamma) \to \mathbb{C} \rightsquigarrow \hat{\hat{f}} : \mathcal{C}(\Gamma) \to \mathbb{C}$$

$$\hat{f}(\mathcal{X}) = \sum_{\mathcal{C} \in \mathcal{C}(\Gamma)} \frac{\mathcal{X}(\mathcal{C}) f(\mathcal{C}) |\mathcal{C}|}{\mathcal{X}(1)} \qquad \hat{\hat{f}}(\mathcal{C}) = \sum_{\chi \in \mathrm{Irr}(\Gamma)} \frac{\chi(\mathcal{C}) \hat{f}(\chi) \chi(1)}{|\mathcal{C}|}$$

- Fourier inversion formula: $\hat{\hat{f}}(h) = |\Gamma| f(h^{-1})$

  Fourier of convolution: $\widehat{f_1 \star f_2} = \hat{f}_1 \cdot \hat{f}_2$

- $\#\{a_1 \in \mathcal{C}_1, \ldots, a_k \in \mathcal{C}_k | a_1 a_2 \cdots a_k = 1\} = 1_{\mathcal{C}_1} \star \cdots \star 1_{\mathcal{C}_k}(1) =$

  $\frac{1}{|\Gamma|} \widehat{\hat{1}_{\mathcal{C}_1} \cdots \hat{1}_{\mathcal{C}_k}}(1) = \sum_{\chi \in \mathrm{Irr}(\Gamma)} \frac{\chi(1)^2}{|\Gamma|} \prod_{i=1}^k \frac{\chi(\mathcal{C}_i) |\mathcal{C}_i|}{\chi(1)}$
- first proved in [Frobenius 1896], $k = 2$ Frobenius orthogonality

Arithmetic harmonic analysis for $\mathrm{GL}(n, \mathbb{F}_q)$

- $\Gamma = \mathrm{GL}(n, \mathbb{F}_q)$
- character table of $\mathrm{GL}(n, \mathbb{F}_q)$ was calculated by

(Jordan, Schur, 1907)	for $n = 2$
(Steinberg, 1951)	for $n = 3, 4$
(Green, 1955)	for all n

- $\mathcal{C}'_1, \ldots, \mathcal{C}'_k \subset \mathrm{GL}(n, \mathbb{F}_q)$ generic semisimple (Hausel, Letellier, Villegas, 2007) calculated explicitly

$$\frac{1}{|\mathrm{PGL(n, \mathbb{F}_q)}|} \#\{a_1 \in \mathcal{C}'_1, \ldots, a_k \in \mathcal{C}'_k | a_1 a_2 \cdots a_k = 1\} =$$

$$= \frac{1}{|\mathrm{PGL(n, \mathbb{F}_q)}|} |\mathrm{Hom}(\pi_1(\mathbb{P}^1_{\mathcal{C}}), \mathrm{GL}(n, F_q))| =$$

$$= \#\{\mathcal{M}_{\mathrm{B}}(\mathbb{F}_q)\},$$

$\tilde{\mathcal{C}}_1, \ldots, \tilde{\mathcal{C}}_k \subset \mathrm{GL}(n, \mathbb{C})$ generic semisimple, $\tilde{\mathcal{C}}_i(\mathbb{F}_q) = \mathcal{C}'_i$

$$\mathcal{M}_{\mathrm{B}} = \{(A_1, \ldots, A_k)\, A_i \in \tilde{\mathcal{C}}_i | A_1 \cdots A_k = Id\} // \mathrm{GL(n, \mathbb{C})}$$

character variety of $\mathbb{P}^1_{\tilde{\mathcal{C}}}$, as a group-valued symplectic quotient

$|GL_3(\mathbb{F}_q)| = (q^3-1)(q^3-q)(q^3-q^2) = q^3(q-1)^3(q+1)(q^2+q+1)$

Tables des caractères de $GL_3(\mathbb{F}_q)$

$|T^F| = (q-1)^3$ | $|W(T)^F| = 6$

$|T_s^F| = (q^2-1)(q-1)$ | $|W(T_s)^F| = 2$

$|T_\sigma^F| = q^3-1$ | $|W(T_\sigma)^F| = 3$

$\left|C_G\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}^F\right| = q^3(q-1)^2$

$\left|C_G\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}^F\right| = q^2(q-1)$

$\left|Cl_G\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}\right| = (q^3-1)(q+1)$

$\left|Cl_{G^F}\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}\right| = (q^3-1)(q^3-q)$

		$\begin{pmatrix}a&0&0\\0&a&0\\0&0&a\end{pmatrix}$ $a\in\mathbb{F}_q^\times$	$\begin{pmatrix}a&0&0\\0&a&0\\0&0&b\end{pmatrix}$ $a\neq b,\ a,b\in\mathbb{F}_q^\times$	$\begin{pmatrix}a&0&0\\0&b&0\\0&0&c\end{pmatrix}$ a,b,c 2 à 2 $\neq$ $\in\mathbb{F}_q^\times$	$\begin{pmatrix}t_1^q&0&0\\0&t_1&0\\0&0&t_2\end{pmatrix}$ $t_1\in\mathbb{F}_{q^2}-\mathbb{F}_q$ $t_2\in\mathbb{F}_q^\times$	$\begin{pmatrix}t_1&0&0\\0&t_1^q&0\\0&0&t_1^{q^2}\end{pmatrix}$ $t_1\in\mathbb{F}_{q^3}-\mathbb{F}_{q^2}$	$\begin{pmatrix}a&0&0\\0&a&1\\0&0&a\end{pmatrix}$ $a\in\mathbb{F}_q^\times$	$\begin{pmatrix}a&1&0\\0&a&1\\0&0&a\end{pmatrix}$ $a\in\mathbb{F}_q^\times$	$\begin{pmatrix}a&0&0\\0&b&1\\0&0&b\end{pmatrix}$ $a\neq b\in\mathbb{F}_q^\times$
	Nombre de classes de ce type	$q-1$	$(q-1)(q-2)$	$\frac{(q-1)(q-2)(q-3)}{6}$	$\frac{q(q-1)(q-1)}{6}$	$\frac{q(q^2-q)}{6}$	$q-1$	$q-1$	$(q-1)(q-2)$
	Cardinal des classes	1	$q^2(q^2+q+1)$	$q^3(q+1)(q^2+q+1)$	$q^2(q^2+q+1)(q^2-1)$	$(q^3-q)(q^3-q^2)$	$(q^3-1)(q+1)$	$(q^3-1)(q^3-q)$	$q^2(q^3-1)(q+1)$
6	$R_T^G(\alpha,\beta,\gamma)$ $\alpha,\beta,\gamma\in Irr(\mathbb{F}_q^\times)$ (α,β,γ) 2 à 2 $\neq$	$(q+1)(q^2+q+1)$ $\times\alpha(a)\beta(a)\gamma(a)$	$(q+1)(\alpha(a)\beta(a)\gamma(b)$ $+\alpha(b)\beta(a)\gamma(a)$ $+\alpha(a)\beta(b)\gamma(a))$	$\sum_{\sigma\in S_3}\alpha(\sigma a)\beta(\sigma b)\gamma(\sigma c)$	0	0	$(1+2q)$ $\times\alpha(a)\beta(a)\gamma(a)$	$\alpha(a)\beta(a)\gamma(a)$	$\alpha(a)\beta(b)\gamma(b)$ $+\alpha(b)\beta(a)\gamma(b)$ $+\alpha(b)\beta(b)\gamma(a)$
1	$Id_{G^F}.(\alpha\circ\det)$ $\alpha\in Irr(\mathbb{F}_q^\times)$	$\alpha(a^3)$	$\alpha(a^2b)$	$\alpha(abc)$	$\alpha(t_1{}^F t_1 t_2)$	$\alpha(t_1{}^F t_1{}^{F^2} t_1)$	$\alpha(a^3)$	$\alpha(a^3)$	$\alpha(ab^2)$
3	$St_G.(\alpha\circ\det)$ $\alpha\in Irr(\mathbb{F}_q^\times)$	$q^3\alpha(a^3)$	$q\alpha(a^2b)$	$\alpha(abc)$	$-\alpha(t_1{}^F t_1 t_2)$	$\alpha(t_1{}^F t_1{}^{F^2} t_1)$	0	0	0
	$R_{T_\sigma}^G(w)$, $w\neq w^q$ $w\in Irr(\mathbb{F}_{q^3}^\times)$	$(q^2-1)(q-1)w(a)$	0	0	0	$w(t_1)+w(t_1^q)$ $+w(t_1^{q^2})$	$(1-q)w(a)$	$w(a)$	0
7	$-R_{T_s}^G(w,\alpha)$ $w\in Irr(\mathbb{F}_{q^2}^\times)$, $\neq w^q$ $\alpha\in Irr(\mathbb{F}_q^\times)$	$(q^3-1)w(a)\alpha(a)$	$(q-1)w(a)\alpha(b)$	0	$-w(t_1)\alpha(t_2)$ $-w(t_1^q)\alpha(t_2)$	0	$-w(a)\alpha(a)$	$-w(a)\alpha(a)$	$-w(b)\alpha(b)$
2	$R_\theta.(\alpha\circ\det)$ $\alpha\in Irr(\mathbb{F}_q^\times)$	$q(q+1)\alpha(a^3)$	$(q+1)\alpha(a^2b)$	$2\alpha(abc)$	0	$-\alpha(t_1{}^F t_1{}^{F^2} t_1)$	$q\alpha(a^3)$	0	$\alpha(ab^2)$
4	$R_{C_G(s)}^G(Id.(\alpha\circ\det,\beta))$ $\alpha\neq\beta\in Irr(\mathbb{F}_q^\times)$	$(q^2+q+1)\alpha(a^2)\beta(a)$	$\alpha(a^2)\beta(b)+(q+1)$ $\times\alpha(a)\alpha(b)\beta(a)$	$\alpha(ab)\beta(c)$ $+[\alpha(ac)\beta(b)+\alpha(bc)\beta(a)]$	$\alpha(t_1^{q+1})\beta(t_2)$	0	$(q+1)\alpha(a^2)\beta(a)$	$\alpha(a^2)\beta(a)$	$\alpha(ab)\beta(b)$ $+\alpha(b^2)\beta(a)$
5	$R_{C_G(s)}^G(St_{C_G(s)}.(\alpha\circ\det,\beta))$ $\alpha\neq\beta\in Irr(\mathbb{F}_q^\times)$	$q(q^2+q+1)$ $\times\alpha(a^2)\beta(a)$	$q\alpha(a^2)\beta(b)+(q+1)$ $\times\alpha(a)\alpha(b)\beta(a)$	$\alpha(ab)\beta(c)$ $+\alpha(ac)\beta(b)$ $+\alpha(bc)\beta(a)$	$-\alpha(t_1^{q+1})\beta(t_2)$	0	$q.\alpha(a^2)\beta(a)$	0	$\alpha(b^2)\beta(a)$

Example

$n = 3$, $\tilde{\mathcal{C}}_i \subset \mathrm{GL}(3, \mathbb{F}_q)$ regular semisimple:

$$\#\{\mathcal{M}_{\mathrm{B}}(\mathbb{F}_q)\} = \sum_{\chi \in \mathrm{Irr}(\Gamma)} \frac{\chi(1)^2}{|\Gamma|} \prod_{i=1}^{k} \frac{\chi(\tilde{\mathcal{C}}_i)|\tilde{\mathcal{C}}_i|}{\chi(1)} =$$

$$= \frac{\left((q+1)\left(q^2+q+1\right)\right)^k}{\left(q^3-1\right)^2\left(q^2-1\right)^2} - \frac{\left(3\,q^2\,(q+1)\right)^k}{q^4\left(q^2-1\right)^2(q-1)^2}$$

$$+1/3\,\frac{\left(6\,q^3\right)^k}{q^6\,(q-1)^4} + \frac{\left(2\,q^2\left(q^2+q+1\right)\right)^k}{q^4\left(q^3-1\right)^2(q-1)^2}$$

$$+\frac{\left(q^3\,(q+1)\left(q^2+\ q+1\right)\right)^k}{q^6\left(q^3-1\right)^2\left(q^2-1\right)^2} - \frac{\left(3\,q^3\,(q+1)\right)^k}{q^6\left(q^2-1\right)^2(q-1)^2} = E(\mathcal{M}_{\mathrm{B}}, q).$$

e.g. $k = 3 \rightsquigarrow E(\mathcal{M}_{\mathrm{B}}; q) = q^2 + 6q + 1$,

$\dim(\mathcal{M}_{\mathrm{B}}) = 2$ and $\mathcal{M}_{\mathrm{B}}$ affine $\Rightarrow$ MHS is not pure on $H^*(\mathcal{M}_{\mathrm{B}})$

What is $H(\mathcal{M}_{\mathrm{B}}; q, t)$? What is $PH(\mathcal{M}_{\mathrm{B}}; q)$?

Conjecture (Hausel, 2004)

When $n=3$, $\tilde{\mathcal{C}}_i$ *regular semisimple,* $h^{p,p;d} = h^{N-p,N-p;d+N-2p}$ *for*

$$H(\mathcal{M}_{\mathrm{B}},q,t) = \sum h^{p,p;d} q^p t^d = \frac{\left(\left(qt^2+1\right)\left(q^2t^4+qt^2+1\right)\right)^k}{\left(q^3t^6-1\right)\left(q^3t^4-1\right)\left(q^2t^4-1\right)\left(q^2t^2-1\right)}$$
$$-\frac{\left(3\,q^2t^4\left(qt^2+1\right)\right)^k}{q^4t^8\left(q^2t^4-1\right)\left(q^2t^2-1\right)\left(qt^2-1\right)\left(q-1\right)}$$
$$+1/3\,\frac{6^k\left(qt^2\right)^{3\,k}}{q^6t^{12}\left(qt^2-1\right)^2\left(q-1\right)^2}+\frac{\left(q^2t^4\left(2\,q^2t^2+qt^2+q+2\right)\right)^k}{q^4t^8\left(q^3t^4-1\right)\left(q^3t^2-1\right)\left(qt^2-1\right)\left(q-1\right)}$$
$$+\frac{\left(q^3t^6\left(q+1\right)\left(q^2+q+1\right)\right)^k}{q^6t^{12}\left(q^3t^2-1\right)\left(q^3-1\right)\left(q^2t^2-1\right)\left(q^2-1\right)}$$
$$-\frac{\left(3\,q^3t^6\left(q+1\right)\right)^k}{q^6t^{12}\left(q^2t^2-1\right)\left(q^2-1\right)\left(qt^2-1\right)\left(q-1\right)},$$

e.g. $k=3 \rightsquigarrow H(\mathcal{M}_{\mathrm{B}};q,t) = 1+6qt^2+q^2t^2$

note: $PH(\mathcal{M}_{\mathrm{B}};t^2) = P(Q,t) \Leftarrow$ "purity conjecture"

Master Conjecture

Conjecture (Hausel-Letellier-Villegas, 2007)

$\mu = (\mu^i)_{i=1}^k \in \mathcal{P}(n)^{\{1,\dots,k\}}$ *type of the conjugacy classes* $(\mathcal{C}_i)_{i=1}^k$

$$\sum_{p,k} h^{p,p;k}(\mathcal{M}_{\mathrm{B}}^{\mu}) q^p t^k = (t\sqrt{q})^{d_\mu}(q-1)(1-\tfrac{1}{qt^2}) \cdot$$
$$\cdot \left\langle \mathrm{Log}\left(\sum_{\lambda\in\mathcal{P}} \left(\prod_{i=1}^k \tilde{H}_\lambda(\mathbf{x}_i; q, \tfrac{1}{qt^2})\right) \mathcal{H}_\lambda(q, \tfrac{1}{qt^2})\right), h_\mu \right\rangle,$$

where $\tilde{H}_\lambda(\mathbf{x}_i; q, \frac{1}{qt^2})$ *are the Macdonald symmetric functions.*

Theorem (Hausel-Letellier-Villegas, 2007)

- *The Master Conjecture is true when specialized to* $t = -1$ *giving a formula for* $E(\mathcal{M}_{\mathrm{B}}; q) = \#\{\mathcal{M}_{\mathrm{B}}(\mathbb{F}_q)\}$.
- *Taking the pure part of the Master Conjecture gives the Poincaré polynomial of the quiver variety* Q, *consistently with the purity conjecture.*
- *When* $k = 2$ *the Master Conjecture is true and reduces to the Cauchy identity for Macdonald polynomials; thus it is a deformation of Frobenius orthogonality for* $\mathrm{GL}(n, \mathbb{F}_q)$.