Arithmetic Harmonic Analysis on holomorphic symplectic quotients

Based on joint work with Letellier and Villegas
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Frobenius on group characters

- Frobenius, F.G.: Über Gruppencharacktere (1896):

  " I shall develop the concept [of character for arbitrary finite groups] here in the belief that through its introduction, group theory will be substantially enriched. "

- After proving the orthogonality relations ($k = 2$ below) Frobenius' first application was:

Theorem (Frobenius 1896)

Let $\mathcal{C}_1, \ldots, \mathcal{C}_k \subset \Gamma$ be conjugacy classes in a finite group Γ then

$$\#\{a_1 \in \mathcal{C}_1, \ldots, a_k \in \mathcal{C}_k | a_1 a_2 \cdots a_k = 1\} = \\ = \sum_{\chi \in \mathrm{Irr}(\Gamma)} \frac{\chi(1)^2}{|\Gamma|} \prod_{i=1}^{k} \frac{\chi(\mathcal{C}_i)|\mathcal{C}_i|}{\chi(1)}$$

Harmonic analysis on finite groups

- Γ finite group; the *convolution* of $f_1, \ldots, f_k : \Gamma \to \mathbb{C}$ is defined:

$$f_1 \star f_2 \star \cdots \star f_k(h) = \sum_{\substack{g_1, g_2, \ldots, g_k \in \Gamma \\ g_1 g_2 \ldots g_k = h}} f_1(g_1) f_2(g_2) \ldots f_k(g_k)$$

- Fourier transform:

$$f : \mathcal{C}(\Gamma) \to \mathbb{C} \rightsquigarrow \hat{f} : \mathrm{Irr}(\Gamma) \to \mathbb{C} \qquad \hat{f} : \mathrm{Irr}(\Gamma) \to \mathbb{C} \rightsquigarrow \hat{\hat{f}} : \mathcal{C}(\Gamma) \to \mathbb{C}$$

$$\hat{f}(\mathcal{X}) = \sum_{\mathcal{C} \in \mathcal{C}(\Gamma)} \frac{\mathcal{X}(\mathcal{C}) f(\mathcal{C}) |\mathcal{C}|}{\mathcal{X}(1)} \qquad \hat{\hat{f}}(\mathcal{C}) = \sum_{\chi \in \mathrm{Irr}(\Gamma)} \chi(\mathcal{C}) \hat{f}(\chi) \chi(1)$$

- Fourier inversion formula: $\hat{\hat{f}}(h) = |\Gamma| f(h^{-1})$

  Fourier of convolution: $\widehat{f_1 \star f_2} = \hat{f}_1 \cdot \hat{f}_2$

- $\#\{a_1 \in \mathcal{C}_1, \ldots, a_k \in \mathcal{C}_k | a_1 a_2 \cdots a_k = 1\} = 1_{\mathcal{C}_1} \star \cdots \star 1_{\mathcal{C}_k}(1) =$

  $\frac{1}{|\Gamma|} \widehat{\hat{1}_{\mathcal{C}_1} \cdots \hat{1}_{\mathcal{C}_k}}(1) = \sum_{\chi \in \mathrm{Irr}(\Gamma)} \frac{\chi(1)^2}{|\Gamma|} \prod_{i=1}^{k} \frac{\chi(\mathcal{C}_i)|\mathcal{C}_i|}{\chi(1)}$

Arithmetic harmonic analysis for $\mathrm{GL}_n(\mathbb{F}_q)$

- $\Gamma = \mathrm{GL}_n(\mathbb{F}_q)$
- character table of $\mathrm{GL}_n(\mathbb{F}_q)$ was calculated by

  (Jordan, Schur, 1907) for $n = 2$

  (Steinberg, 1951) for $n = 3, 4$

  (Green, 1955) for all n
- $\mathcal{C}'_1, \dots, \mathcal{C}'_k \subset \mathrm{GL}_n(\mathbb{F}_q)$ generic semisimple

  (Hausel, Letellier, Villegas, 2008) calculated explicitly

$$\sum_{\chi \in \mathrm{Irr}(\mathrm{GL}_n(\mathbb{F}_q))} \frac{\chi(1)^2}{|\mathrm{GL}_n(\mathbb{F}_q)|} \prod_{i=1}^k \frac{\chi(\mathcal{C}'_i)|\mathcal{C}'_i|}{\chi(1)} =$$
$$\#\{a_1 \in \mathcal{C}'_1, \dots, a_k \in \mathcal{C}'_k | a_1 a_2 \cdots a_k = 1\} =$$
$$= |\mathrm{Hom}(\pi_1(\mathbb{P}^1_{\mathcal{C}}), \mathrm{GL}(n, F_q))| = |\mathrm{PGL}(\mathrm{n}, \mathbb{F}_\mathrm{q})| \#\{\mathcal{M}_\mathrm{B}(\mathbb{F}_q)\},$$

$\tilde{\mathcal{C}}_1, \dots, \tilde{\mathcal{C}}_k \subset \mathrm{GL}(n, \mathbb{C})$ generic semisimple, $\tilde{\mathcal{C}}_i(\mathbb{F}_q) = \mathcal{C}'_i$

$$\mathcal{M}_\mathrm{B} = \{(A_1, \dots, A_k)\, A_i \in \tilde{\mathcal{C}}_i | A_1 \cdots A_k = Id\} // \mathrm{GL}(\mathrm{n}, \mathbb{C})$$

character variety of $\mathbb{P}^1_{\tilde{\mathcal{C}}}$, as a group-valued symplectic quotient

$|GL_3(\mathbb{F}_q)| = (q^3-1)(q^3-q)(q^3-q^2) = q^3(q-1)^3(q+1)(q^2+q+1)$

Tables des caractères de $GL_3(\mathbb{F}_q)$

$|T^F| = (q-1)^3$ | $|W(T)^F| = 6$

$|T_s^F| = (q^2-1)(q-1)$ | $|W(T_s)^F| = 2$

$|T_\sigma^F| = q^3-1$ | $|W(T_\sigma)^F| = 3$

$\left|C_G\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}^F\right| = q^3(q-1)^2$

$\left|C_G\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}^F\right| = q^2(q-1)$

$\left|\mathcal{C}_G\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}\right| = (q^3-1)(q+1)$

$\left|\mathcal{C}_{G^F}\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}\right| = (q^3-1)(q^3-q)$

		$\begin{pmatrix}a&0&0\\0&a&0\\0&0&a\end{pmatrix}$ $a\in\mathbb{F}_q^\times$	$\begin{pmatrix}a&0&0\\0&a&0\\0&0&b\end{pmatrix}$ $a\neq b,\ a,b\in\mathbb{F}_q^\times$	$\begin{pmatrix}a&0&0\\0&b&0\\0&0&c\end{pmatrix}$ a,b,c 2 à 2 $\neq$ $\in\mathbb{F}_q^\times$	$\begin{pmatrix}t_1^q&0&0\\0&t_1&0\\0&0&t_2\end{pmatrix}$ $t_1\in\mathbb{F}_{q^2}^\times-\mathbb{F}_q$, $t_2\in\mathbb{F}_q^\times$	$\begin{pmatrix}t_1&0&0\\0&t_1^q&0\\0&0&t_1^{q^2}\end{pmatrix}$ $t_1\in\mathbb{F}_{q^3}^\times-\mathbb{F}_{q^2}$	$\begin{pmatrix}a&0&0\\0&a&1\\0&0&a\end{pmatrix}$ $a\in\mathbb{F}_q^\times$	$\begin{pmatrix}a&1&0\\0&a&1\\0&0&a\end{pmatrix}$ $a\in\mathbb{F}_q^\times$	$\begin{pmatrix}a&0&0\\0&b&1\\0&0&b\end{pmatrix}$ $a\neq b\in\mathbb{F}_q^\times$
	Nombre de classes de ce type	$q-1$	$(q-1)(q-2)$	$\frac{(q-1)(q-2)(q-3)}{6}$	$\frac{q(q-1)(q-1)}{6}$	$\frac{q(q^2-q)}{6}$	$q-1$	$q-1$	$(q-1)(q-2)$
	Cardinal des classes	1	$q^2(q^2+q+1)$	$q^3(q+1)(q^2+q+1)$	$q^2(q^2+q+1)(q^2-1)$	$(q^3-q)(q^3-q^2)$	$(q^3-1)(q+1)$	$(q^3-1)(q^3-q)$	$q^2(q^3-1)(q+1)$
6	$R_T^G(\alpha,\beta,\gamma)$ $\alpha,\beta,\gamma\in Irr(\mathbb{F}_q^\times)$ (α,β,γ) 2 à 2 $\neq$	$(q+1)(q^2+q+1)\times\alpha(a)\beta(a)\gamma(a)$	$(q+1)(\alpha(a)\beta(a)\gamma(b)$ $+\alpha(b)\beta(a)\gamma(a)$ $+\alpha(a)\beta(b)\gamma(a))$	$\sum_{\sigma\in S_3}\alpha(\sigma a)\beta(\sigma b)\gamma(\sigma c)$	0	0	$(1+2q)\times\alpha(a)\beta(a)\gamma(a)$	$\alpha(a)\beta(a)\gamma(a)$	$\alpha(a)\beta(b)\gamma(b)$ $+\alpha(b)\beta(a)\gamma(b)$ $+\alpha(b)\beta(b)\gamma(a)$
1	$Id_{G^F}.(\alpha\circ\det)$ $\alpha\in Irr(\mathbb{F}_q^\times)$	$\alpha(a^3)$	$\alpha(a^2b)$	$\alpha(abc)$	$\alpha(t_1^F t_1 t_2)$	$\alpha(t_1^F t_1^{F^2} t_1)$	$\alpha(a^3)$	$\alpha(a^3)$	$\alpha(ab^2)$
3	$St_G.(\alpha\circ\det)$ $\alpha\in Irr(\mathbb{F}_q^\times)$	$q^3\alpha(a^3)$	$q\alpha(a^2b)$	$\alpha(abc)$	$-\alpha(t_1^F t_1 t_2)$	$\alpha(t_1^F t_1^{F^2} t_1)$	0	0	0
	$R_{T_\sigma}^G(w)$, $w\neq w^q$ $w\in Irr(\mathbb{F}_{q^3}^\times)$	$(q^2-1)(q-1)w(a)$	0	0	0	$w(t_1)+w(t_1^q)+w(t_1^{q^2})$	$(1-q)w(a)$	$w(a)$	0
7	$-R_{T_s}^G(w,\alpha)$ $w\in Irr(\mathbb{F}_{q^2}^\times)$, $\neq w^q$ $\alpha\in Irr(\mathbb{F}_q^\times)$	$(q^3-1)w(a)\alpha(a)$	$(q-1)w(a)\alpha(b)$	0	$-w(t_1)\alpha(t_2)$ $-w(t_1^q)\alpha(t_2)$	0	$-w(a)\alpha(a)$	$-w(a)\alpha(a)$	$-w(b)\alpha(b)$
2	$R_\theta.(\alpha\circ\det)$ $\alpha\in Irr(\mathbb{F}_q^\times)$	$q(q+1)\alpha(a^3)$	$(q+1)\alpha(a^2b)$	$2\alpha(abc)$	0	$-\alpha(t_1^F t_1^{F^2} t_1)$	$q\alpha(a^3)$	0	$\alpha(ab^2)$
4	$R_{C_G(s)}^G(Id.(\alpha\circ\det,\beta))$ $\alpha\neq\beta\in Irr(\mathbb{F}_q^\times)$	$(q^2+q+1)\alpha(a^2)\beta(a)$	$\alpha(a^2)\beta(b)+(q+1)\times\alpha(a)\alpha(b)\beta(a)$	$\alpha(ab)\beta(c)+[\alpha(ac)\beta(b)+\alpha(bc)\beta(a)]$	$\alpha(t_1^{q+1})\beta(t_2)$	0	$(q+1)\alpha(a^2)\beta(a)$	$\alpha(a^2)\beta(a)$	$\alpha(ab)\beta(b)+\alpha(b^2)\beta(a)$
5	$R_{C_G(s)}^G(St_{C_G(s)}.(\alpha\circ\det,\beta))$ $\alpha\neq\beta\in Irr(\mathbb{F}_q^\times)$	$q(q^2+q+1)\times\alpha(a^2)\beta(a)$	$q\alpha(a^2)\beta(b)+(q+1)\times\alpha(a)\alpha(b)\beta(a)$	$\alpha(ab)\beta(c)+\alpha(ac)\beta(b)+\alpha(bc)\beta(a)$	$-\alpha(t_1^{q+1})\beta(t_2)$	0	$q.\alpha(a^2)\beta(a)$	0	$\alpha(b^2)\beta(a)$

Example

$n = 3$, $\mathcal{C}_i' \subset \mathrm{GL}_3(\mathbb{F}_q)$ regular semisimple:

$$\#\{\mathcal{M}_{\mathrm{B}}(\mathbb{F}_q)\} = \sum_{\chi \in \mathrm{Irr}(\mathrm{GL}_3(\mathbb{F}_q))} \frac{\chi(1)^2}{|\mathrm{GL}_3(\mathbb{F}_q)|} \prod_{i=1}^{k} \frac{\chi(\mathcal{C}_i')|\mathcal{C}_i'|}{\chi(1)} =$$

$$= \frac{\left((q+1)\left(q^2+q+1\right)\right)^k}{\left(q^3-1\right)^2\left(q^2-1\right)^2} - \frac{\left(3\,q^2\,(q+1)\right)^k}{q^4\left(q^2-1\right)^2(q-1)^2}$$
$$+ 1/3\,\frac{\left(6\,q^3\right)^k}{q^6\,(q-1)^4} + \frac{\left(2\,q^2\left(q^2+q+1\right)\right)^k}{q^4\left(q^3-1\right)^2(q-1)^2}$$
$$+ \frac{\left(q^3\,(q+1)\left(q^2+q+1\right)\right)^k}{q^6\left(q^3-1\right)^2\left(q^2-1\right)^2} - \frac{\left(3\,q^3\,(q+1)\right)^k}{q^6\left(q^2-1\right)^2(q-1)^2}.$$

e.g. $k = 3 \rightsquigarrow \#\{\mathcal{M}_{\mathrm{B}}(\mathbb{F}_q)\} = q^2 + 6q + 1$

Mixed Hodge Structure of Deligne

- $\bigoplus_{p,q} H^{p,q;k}(M)$ is the associated graded to the weight and Hodge filtrations on the cohomology $H^k(M, \mathbb{C})$ of a complex algebraic variety M
- $h^{p,q;k} = \dim(H^{p,q;k}(M))$, *mixed Hodge numbers*
- $H(M; x, y, t) = \sum_{p,q,k} h^{p,q;k}(M) x^p y^q t^k$, *mixed Hodge polynomial*
- $P(M; t) = H(M; 1, 1, t)$, *Poincaré polynomial*
- $E(M; x, y) = x^n y^n H(1/x, 1/y, -1)$, *E-polynomial* of a smooth variety M.

Arithmetic and topological content of the E-polynomial

Theorem (Katz 2006)

If M is a smooth quasi-projective variety defined over $\mathbb{Z}$ and

$$\#\{M(\mathbb{F}_q)\} = E(q)$$

is a polynomial in q, then $E(M;x,y) = E(xy)$.

- MHS on $H^*(M,\mathbb{C})$ is *pure* if $h^{p,q;k} = 0$ unless $p+q=k \Leftrightarrow$ $H(M;x,y,t) = (xyt^2)^n E(\frac{-1}{xt}, \frac{-1}{yt}) \Rightarrow$ $P(M;t) = H(M;1,1,t) = t^{2n} E(\frac{-1}{t}, \frac{-1}{t})$; examples: smooth projective varieties, $\mathcal{M}_{\mathrm{Dol}}$, $\mathcal{M}_{\mathrm{DR}}$, Nakajima's quiver varieties
- in general the pure part of $H(M;x,y,t)$ is $PH(M;x,y) = \mathrm{Coeff}_{T^0}\left(H(M;xT,yT,tT^{-1})\right)$; which, for a smooth M, is always the image of the cohomology of a smooth compactification

Example

$n = 3$, $\tilde{\mathcal{C}}_i$ regular semisimple ($q = xy$):

$$E(\mathcal{M}_{\mathrm{B}}, q) =$$

$$\frac{\left((q+1)\left(q^2+q+1\right)\right)^k}{\left(q^3-1\right)^2\left(q^2-1\right)^2} - \frac{\left(3\,q^2\,(q+1)\right)^k}{q^4\left(q^2-1\right)^2(q-1)^2}$$

$$+ 1/3\,\frac{\left(6\,q^3\right)^k}{q^6\,(q-1)^4} + \frac{\left(2\,q^2\left(q^2+q+1\right)\right)^k}{q^4\left(q^3-1\right)^2(q-1)^2}$$

$$+ \frac{\left(q^3\,(q+1)\left(q^2+\ q+1\right)\right)^k}{q^6\left(q^3-1\right)^2\left(q^2-1\right)^2} - \frac{\left(3\,q^3\,(q+1)\right)^k}{q^6\left(q^2-1\right)^2(q-1)^2}.$$

e.g. $k = 3 \rightsquigarrow E(\mathcal{M}_{\mathrm{B}}; q) = q^2 + 6q + 1$,

$\dim(\mathcal{M}_{\mathrm{B}}) = 2$ and $\mathcal{M}_{\mathrm{B}}$ affine $\Rightarrow$ MHS is not pure on $H^*(\mathcal{M}_{\mathrm{B}})$

What is $H(\mathcal{M}_{\mathrm{B}}; q, t)$? What is $PH(\mathcal{M}_{\mathrm{B}}; q)$?

Conjecture (Hausel, 2004)

When $n=3$, $\tilde{\mathcal{C}}_i$ *regular semisimple,* $h^{p,p;d} = h^{N-p,N-p;d+N-2p}$ *for*

$$H(\mathcal{M}_{\mathrm{B}},q,t) = \sum h^{p,p;d} q^p t^d = \frac{\left(\left(qt^2+1\right)\left(q^2t^4+qt^2+1\right)\right)^k}{\left(q^3t^6-1\right)\left(q^3t^4-1\right)\left(q^2t^4-1\right)\left(q^2t^2-1\right)}$$
$$-\frac{\left(3\,q^2t^4\left(qt^2+1\right)\right)^k}{q^4t^8\left(q^2t^4-1\right)\left(q^2t^2-1\right)\left(qt^2-1\right)\left(q-1\right)}$$
$$+1/3\,\frac{6^k\left(qt^2\right)^{3\,k}}{q^6t^{12}\left(qt^2-1\right)^2\left(q-1\right)^2}$$
$$+\frac{\left(q^2t^4\left(2\,q^2t^2+qt^2+q+2\right)\right)^k}{q^4t^8\left(q^3t^4-1\right)\left(q^3t^2-1\right)\left(qt^2-1\right)\left(q-1\right)}$$
$$+\frac{\left(q^3t^6\left(q+1\right)\left(q^2+q+1\right)\right)^k}{q^6t^{12}\left(q^3t^2-1\right)\left(q^3-1\right)\left(q^2t^2-1\right)\left(q^2-1\right)}$$
$$-\frac{\left(3\,q^3t^6\left(q+1\right)\right)^k}{q^6t^{12}\left(q^2t^2-1\right)\left(q^2-1\right)\left(qt^2-1\right)\left(q-1\right)},$$

e.g. $k=3 \rightsquigarrow H(\mathcal{M}_{\mathrm{B}};q,t) = 1+6qt^2+q^2t^2$

The Riemann-Hilbert map

- $\mathfrak{g} = \mathfrak{gl}_n$
- $C = \mathbb{P}^1$ with punctures $a_1, \ldots, a_k \in \mathbb{P}^1$
- $\mathcal{C}_i$ semisimple adjoint orbit in $\mathfrak{g}(\mathbb{C})$
- $Q = \{(A_1, \ldots, A_k)\, A_i \in \mathcal{C}_i | A_1 + \cdots + A_k = 0\}//G(\mathbb{C})$, *star-shaped quiver variety*, as holomorphic symplectic quotient
- Q is smooth when $\mathcal{C}_i$ are generic
- "$Q \subset \mathcal{M}_{\mathrm{DR}}$", a point in Q gives the meromorphic flat $GL(n, \mathbb{C})$-connection $\sum A_i \frac{dz}{z-a_i}$ on the trivial bundle on C.
- $\tilde{\mathcal{C}}_i = \exp(2\pi i \mathcal{C}_i) \subset G(\mathbb{C})$ is the corresponding conjugacy class
- the *Riemann-Hilbert monodromy map*

$$\nu_a : Q \to \mathcal{M}_{\mathrm{B}}$$

is given by sending the flat connection to its holonomy.

The purity conjecture

Conjecture (Hausel, 2004)

If $\mathcal{C}_i$ are generic, then $\nu_a^* : PH^*(\mathcal{M}_{\mathrm{B}}) \xrightarrow{\cong} H^*(Q)$

Example

- $n = 3$, $k = 3$, $\mathcal{C}_i$ regular semisimple
- Q is E_6 ALE space,
- $\mathcal{M}_{\mathrm{B}} \cong \mathcal{M}_{\mathrm{Dol}}$ elliptic fibration with singular fibre of type $\hat{E}_6$.
- $P(Q; t) = 1 + 6t^2$
- $H(\mathcal{M}_{\mathrm{B}}; q, t) = 1 + 6qt^2 + q^2t^2$

$$\Downarrow$$

Conjecture is true in this case

Fourier transform for Q

- $\Psi : \mathbb{F}_q \to \mathbb{C}$ non-trivial additive character
- $f : \mathfrak{g}(\mathbb{F}_q) \to \mathbb{C}$ its Fourier transform $\hat{f} : \mathfrak{g}^*(\mathbb{F}_q) \to \mathbb{C}$

$$\hat{f}(Y) := \sum_{X \in \mathfrak{g}(\mathbb{F}_q)} f(X)\Psi(\langle X, Y \rangle).$$

- $1_{\mathcal{C}_i} : \mathfrak{g}(\mathbb{F}_q) \to \mathbb{C}$ characteristic function of $\mathcal{C}_i \subset \mathfrak{g}(\mathbb{F}_q)$

Theorem

$$|\mathrm{PGL(n,\mathbb{F}_q)}|\#\{\mathrm{Q(\mathbb{F}_q)}\} =$$
$$\#\{a_1 \in \mathcal{C}_1, \ldots, a_k \in \mathcal{C}_k | a_1 + a_2 + \cdots + a_k = 0\} =$$
$$1_{\mathcal{C}_1} \star \cdots \star 1_{\mathcal{C}_k}(0) = \widehat{1_{\mathcal{C}_1} \cdots 1_{\mathcal{C}_k}}(0) =$$
$$\frac{1}{|\mathfrak{gl}_n(\mathbb{F}_q)|} \sum_{X \in \mathfrak{g}(\mathbb{F}_q)} \hat{1}_{\mathcal{C}_1}(X) \cdot \cdots \cdot \hat{1}_{\mathcal{C}_k}(X)$$

To HAUSEL

EMMANUEL LETELLIER:

Character table of $Gl_3(\mathbb{F}_q)$

	$\begin{pmatrix} a&0&0\\0&a&0\\0&0&a\end{pmatrix}$ $a \in \mathbb{F}_q$	$\begin{pmatrix} a&0&0\\0&a&0\\0&0&b\end{pmatrix}$ $a \neq b$, $a,b \in \mathbb{F}_q$	$\begin{pmatrix} a&0&0\\0&b&0\\0&0&c\end{pmatrix}$ $a,b,c \in \mathbb{F}_q$, $a\neq b, a\neq c, b\neq c$	$\begin{pmatrix} t_1^q&0&0\\0&t_1&0\\0&0&t_2\end{pmatrix}$ $t_1 \in \mathbb{F}_{q^2}-\mathbb{F}_q$, $t_2 \in \mathbb{F}_q$	$\begin{pmatrix} t_1&0&0\\0&t_1^q&0\\0&0&t_1^{q^2}\end{pmatrix}$ $t_1 \in \mathbb{F}_{q^3}-\mathbb{F}_{q^2}$	$\begin{pmatrix} a&0&0\\0&a&1\\0&0&a\end{pmatrix}$ $a \in \mathbb{F}_q$	$\begin{pmatrix} a&1&0\\0&a&1\\0&0&a\end{pmatrix}$ $a \in \mathbb{F}_q$	$\begin{pmatrix} a&0&0\\0&b&1\\0&0&b\end{pmatrix}$ $a\neq b$, $b,a \in \mathbb{F}_q$
$\mathcal{F}\begin{pmatrix} \alpha&0&0\\0&\alpha&0\\0&0&\alpha\end{pmatrix}$ $\alpha \in \mathbb{F}_q$	$q^{-9/2}\psi(a\alpha)^3$	$q^{-9/2}\psi(a\alpha)^2\psi(b\alpha)$	$q^{-9/2}\psi(a\alpha+b\alpha+c\alpha)$	$q^{-9/2}\psi(\alpha(t_1+t_1^q))\psi(\alpha t_2)$	$q^{-9/2}\psi(\alpha(t_1+t_1^q+t_1^{q^2}))$	$q^{-9/2}\psi(a\alpha)^3$	$q^{-9/2}\psi(a\alpha)^3$	$q^{-9/2}\psi(a\alpha)\psi(b\alpha)$
$\mathcal{F}\begin{pmatrix} \alpha&0&0\\0&\alpha&0\\0&0&\beta\end{pmatrix}$ $\alpha\neq\beta$, $\alpha,\beta \in \mathbb{F}_q$	$q^{-5/2}(q^2+q+1)\times \psi(a\alpha)^2\psi(b\beta)$	$q^{-5/2}\psi(\alpha a)^2\psi(\beta b) + (q+1)\psi(\alpha b)\psi(\alpha a)\times\psi(\beta a)$	$q^{-5/2}[\psi(\alpha a+\alpha b+\beta c) + \psi(\alpha a+\alpha c+\beta b) + \psi(\alpha b+\alpha c+\beta a)]$	$q^{-5/2}\psi(\alpha(t_1+t_1^q))\times \psi(\beta t_2)$	0	$q^{-5/2}(1+q)$ $\psi(\alpha a)^2\psi(a\beta)$	$q^{-5/2}\psi(\alpha a)^2\psi(a\beta)$	$q^{-5/2}[\psi(\alpha b)^2\psi(\beta a) + \psi(\alpha b)\psi(\alpha a)\psi(a\beta)$
$\mathcal{F}\begin{pmatrix} \alpha&0&0\\0&\beta&0\\0&0&\gamma\end{pmatrix}$ $\alpha,\beta,\gamma \in \mathbb{F}_q$, $\alpha\neq\beta, \alpha\neq\gamma, \beta\neq\gamma$	$q^{-3/2}(q+1)(q^2+q+1)$ $\psi(\alpha a)\psi(\beta a)\psi(\gamma a)$	$q^{-3/2}(q+1)\times[\psi(\alpha a)\psi(\beta a)\psi(\gamma b) + \psi(\alpha a)\psi(\beta b)\psi(\gamma a) + \psi(\alpha b)\psi(\beta a)\psi(\gamma a)]$	$q^{-3/2}\sum_{\sigma\in S_3}\psi(\alpha\sigma(a))\psi(\beta\sigma(b))\psi(\gamma\sigma(c))$	0	0	$q^{-3/2}(1+2q)$ $\psi(\alpha a)\psi(\beta a)\times\psi(\gamma a)$	$q^{-3/2}$ $\psi(\alpha a)\psi(\beta a)\times\psi(\gamma a)$	$q^{-3/2}[\psi(\alpha a)\psi(\beta b)\psi($ $+\psi(\alpha b)\psi(\beta b)\psi($ $+\psi(\alpha b)\psi(\beta a)\psi($
$\mathcal{F}\begin{pmatrix} r_1^q&0&0\\0&r_1&0\\0&0&r_2\end{pmatrix}$ $r_1 \in \mathbb{F}_{q^2}-\mathbb{F}_q$, $r_2 \in \mathbb{F}_q$	$q^{-3/2}(q^2-1)$ $\psi(a(r_1+r_1^q))$ $\psi(ar_2)$	$q^{-3/2}(q-1)$ $\psi(a(r_1+r_1^q))$ $\psi(br_2)$	0	$-q^{-3/2}[\psi(r_1^q t_1^q+r_1t_1)\psi(r_2t_2)+\psi(r_2t_2)\psi(r_1t_1^q+r_1^qt_1)]$	0	$-q^{-3/2}\psi(ar_2)\times\psi(a(r_1+r_1^q))$	$-q^{-3/2}\psi(ar_2)\times\psi(a(r_1+r_1^q))$	$-q^{-3/2}\psi(br_2)\times\psi(a(r_1+r_1^q))$
$\mathcal{F}\begin{pmatrix} r&0&0\\0&r^q&0\\0&0&r^{q^2}\end{pmatrix}$ $r \in \mathbb{F}_{q^3}-\mathbb{F}_{q^2}$	$q^{-3/2}(q^2-1)(q-1)$ $\psi(a(r+r^q+r^{q^2}))$	0	0	0	$\psi(rt_1+(rt_1)^q+(rt_1)^{q^2}) + \psi(rt_1^q+r^qt_1^{q^2}+r^{q^2}t_1) + \psi(rt_1^{q^2}+r^qt_1+r^{q^2}t_1^q)$	$q^{-3/2}(1-q)$ $\psi(a(r+r^q+r^{q^2}))$	$q^{-3/2}$ $\psi(a(r+r^q+r^{q^2}))$	0
$\mathcal{F}\begin{pmatrix} \alpha&0&0\\0&\alpha&1\\0&0&\alpha\end{pmatrix}$ $\alpha \in \mathbb{F}_q$	$q^{-3/2}(q^2-1)(q+1)\times\psi(a\alpha)^3$	$q^{-9/2}(q^2-1)(q+1)$ $\psi(a\alpha)^2\psi(\alpha b)$	$q^{-9/2}(2q+1)(q-1)$ $\psi(\alpha a)\psi(\alpha b)\psi(\alpha c)$	$-q^{-9/2}(q+1)$ $\psi(\alpha(t_1+t_1^q))\psi(\alpha t_2)$	$-q^{-9/2}(q^2+q+1)$ $\psi(\alpha(t_1+t_1^q+t_1^{q^2}))$	$q^{-3/2}(q^2-q-1)$ $\psi(\alpha a)^3$	$-q^{-9/2}(q+1)$ $\psi(\alpha a)^3$	$q^{-9/2}(q^2-q-1)$ $\psi(\alpha a)\psi(\alpha b)^2$
$\mathcal{F}\begin{pmatrix} \alpha&1&0\\0&\alpha&1\\0&0&\alpha\end{pmatrix}$ $\alpha \in \mathbb{F}_q$	$q^{-7/2}(q^3-1)(q^2-1)\times\psi(a\alpha)^3$	$q^{-7/2}(q-1)^2(q+1)$ $\psi(\alpha b)\psi(\alpha a)^2$	$q^{-7/2}(q-1)^2$ $\psi(\alpha a)\psi(\alpha b)\psi(\alpha c)$	$-q^{-7/2}(q^2-1)$ $\psi(\alpha(t_1+t_1^q))\psi(\alpha t_2)$	$q^{-7/2}(q^2+q+1)$ $\psi(\alpha(t_1+t_1^q+t_1^{q^2}))$	$q^{-7/2}(1-q^2)$ $\psi(\alpha a)^3$	$q^{-7/2}$ $\psi(\alpha a)^3$	$q^{-7/2}(1-q)$ $\psi(\alpha a)\psi(\alpha b)^2$
$\mathcal{F}\begin{pmatrix} \alpha&0&0\\0&\beta&1\\0&0&\beta\end{pmatrix}$ $\alpha\neq\beta$, $\alpha,\beta \in \mathbb{F}_q$	$q^{-5/2}(q^4+q^3-q-1)$ $\psi(\alpha a)\psi(\beta a)^2$	$q^{-5/2}(q^2-1)\times[\psi(\alpha b)\psi(\beta a)^2 + \psi(\alpha a)\psi(\beta a)\psi(\beta b)]$	$q^{-5/2}(q-1)$ $[\psi(\alpha a+\beta b+\beta c) + \psi(\alpha b+\beta c+\beta a) + \psi(\alpha c+\beta a+\beta b)]$	$-q^{-5/2}(q+1)$ $\psi(\beta(t_1+t_1^q))$ $\psi(\alpha t_2)$	0	$q^{-5/2}(q^2-q-1)$ $\psi(\beta a)^2\psi(\alpha a)$	$q^{-5/2}$ $\psi(\beta a)^2\psi(\alpha a)$	$-q^{-5/2}[\psi(\beta b)^2\psi(\alpha$ $(1-q)\psi(\alpha b)\psi(\beta a)\psi$

Theorem

Letellier's character table for $\mathfrak{gl}_3(\mathbb{F}_q)$ implies that when $n = 3$ and all adjoint orbits $\mathcal{C}_i$ are regular semi-simple

$$P(Q;t) = \frac{\left(\left(t^2+1\right)\left(t^4+t^2+1\right)\right)^k}{\left(t^6-1\right)\left(t^4-1\right)} - \frac{\left(3\,t^4\left(t^2+1\right)\right)^k}{t^8\left(t^4-1\right)\left(t^2-1\right)} + 1/3\,\frac{6^k\left(t^2\right)^{3\,k}}{t^{12}\left(t^2-1\right)^2} - \frac{\left(t^4\left(t^2+2\right)\right)^k}{t^8\left(t^2-1\right)} + t^{6k-12} + \frac{\left(3\,t^6\right)^k}{t^{12}\left(t^2-1\right)}$$

which agrees with the pure part of the conjectured mixed Hodge polynomial of the corresponding $\mathcal{M}_{\mathrm{B}}$.

Master Conjecture

Conjecture (Hausel-Letellier-Villegas, 2008)

$\mu = (\mu^i)_{i=1}^k \in \mathcal{P}(n)^{\{1,\dots,k\}}$ *type of the conjugacy classes* $(\mathcal{C}_i)_{i=1}^k$

$$\sum_{p,k} h^{p,p;k}(\mathcal{M}_{\mathrm{B}}^{\mu}) q^p t^k = (t\sqrt{q})^{d_\mu}(q-1)(1-\tfrac{1}{qt^2}) \cdot$$
$$\cdot \left\langle \mathrm{Log}\left(\sum_{\lambda\in\mathcal{P}}\left(\prod_{i=1}^k \tilde{H}_\lambda(\mathbf{x}_i;q,\tfrac{1}{qt^2})\right)\mathcal{H}_\lambda(q,\tfrac{1}{qt^2})\right), h_\mu\right\rangle ,$$

where $\tilde{H}_\lambda(\mathbf{x}_i;q,\frac{1}{qt^2})$ *are the Macdonald symmetric functions.*

Theorem (Hausel-Letellier-Villegas, 2008)

- *The Master Conjecture is true when specialized to* $t=-1$ *giving a formula for* $E(\mathcal{M}_{\mathrm{B}};q) = \#\{\mathcal{M}_{\mathrm{B}}(\mathbb{F}_q)\}$.
- *Taking the pure part of the Master Conjecture gives the Poincaré polynomial of the quiver variety* Q*, consistently with the purity conjecture.*
- *When* $k=2$ *the Master Conjecture is true and reduces to the Cauchy identity for Macdonald polynomials; thus it is a deformation of Frobenius' orthogonality for* $\mathrm{GL}_n(\mathbb{F}_q)$.