Arithmetic Harmonic Analysis and cohomology of hyperkähler manifolds

(Motivated by Mirror Symmetry and Langlands Duality in the mixed Hodge structure of character varieties of Riemann surfaces)

Tamás Hausel
University of Texas at Austin

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Motivation: mirror symmetry

- A pair of n dimensional Calabi-Yau manifolds (X, Y) satisfy the topological mirror test if $H^{p,q}(X) = H^{n-p,q}(Y)$

- A pair of n dimensional Calabi-Yau manifolds (X, Y) are Strominger-Yau-Zaslow mirror pairs if they map to the same real n-dimensional manifold B, so that the generic fibers are dual special Lagrangian tori

Diffeomorphic spaces in the non-Abelian Hodge theory of a genus g curve C:

$$\mathcal{M}^d_{\mathsf{Dol}}(GL(n,\mathbb{C})) := \left\{ \begin{array}{c} \text{moduli space of semistable rank } n \\ \text{degree } d \text{ Higgs bundles on } C \end{array} \right\}$$

$$\mathcal{M}^d_{\mathsf{DR}}(GL(n,\mathbb{C})) := \left\{ \begin{array}{c} \text{moduli space of flat } GL(n,\mathbb{C})\text{-connections} \\ \text{on } C \setminus \{p\}, \text{ with holonomy } e^{\frac{2\pi i d}{n}} Id \text{ around } p \end{array} \right\}$$

$$\mathcal{M}^d_{\mathsf{B}}(GL(n,\mathbb{C})) := \{A_1, B_1, \ldots, A_g, B_g \in GL(n,\mathbb{C}) | \\ A_1^{-1}B_1^{-1}A_1B_1 \ldots A_g^{-1}B_g^{-1}A_gB_g = \xi_n Id\}/GL(n,\mathbb{C})$$

Theorem 1 (Hausel–Thaddeus 2003). *In the following diagram*

$$\begin{array}{ccc} \tilde{\mathcal{M}}^d_{\mathsf{Dol}}(PGL(n)) & \longrightarrow & \mathcal{M}^d_{\mathsf{Dol}}(SL(n)) \\ \downarrow\chi_{PGL(n)} & & \downarrow\chi_{SL(n)} \\ \mathcal{H}_{PGL(n)} & \cong & \mathcal{H}_{SL(n)}. \end{array}$$

the generic fibers of the Hitchin maps $\chi_{PGL(n)}$ *and* $\chi_{SL(n)}$ *are dual Abelian varieties.*

$$\Downarrow$$

$\mathcal{M}^d_{\mathsf{DR}}(PGL(n))$ *and* $\mathcal{M}^d_{\mathsf{DR}}(SL(n))$ *satisfy the SYZ construction for a pair of mirror symmetric Calabi-Yau manifolds.*

Topological Mirror Test

Conjecture 1 (Hausel–Thaddeus 2003). *For all $d, e \in \mathbb{Z}$, satisfying $(d,n) = (e,n) = 1$, we have*

$$E_{\mathsf{st}}^{B^e}(x,y;\mathcal{M}_{\mathsf{DR}}^d(SL(n,\mathbb{C}))) = E_{\mathsf{st}}^{\hat{B}^d}(x,y;\mathcal{M}_{\mathsf{DR}}^e(PGL(n,\mathbb{C}))).$$

Conjecture 2 (Hausel–R-Villegas 2004).

$$E_{\mathsf{st}}^{B^e}(x,y,\mathcal{M}_{\mathsf{B}}^d(SL(n,\mathbb{C}))) = E_{\mathsf{st}}^{\hat{B}^d}(x,y,\mathcal{M}_{\mathsf{B}}^e(PGL(n,\mathbb{C}))).$$

Mixed Hodge Structure of Deligne

- $\bigoplus_{p,q} H^{p,q;k}(M)$ is the associated graded to the weight and Hodge filtrations on the cohomology $H^k(M, \mathbb{C})$ of a complex algebraic variety M

- $h^{p,q;k} = \dim(H^{p,q;k}(M))$, the *mixed Hodge numbers*

- $H(M; x, y, t) = \sum_{p,q,k} h^{p,q;k}(M) x^p y^q t^k$, the *mixed Hodge polynomial*

- $P(M; t) = H(M; 1, 1, t)$, the *Poincaré polynomial*

- $E(M; x, y) = x^n y^n H(1/x, 1/y, -1)$, the *E-polynomial* of a smooth variety M.

Topological content of the E-polynomial

- $E(M, 1, 1) = \chi(M)$, the Euler characteristic

- if $H(M; x, y, t) = E(xt, yt)$, $\Leftrightarrow$ MHS on $H^*(M, \mathbb{C})$ is *pure* $\Rightarrow P(M; t) = H(M; 1, 1, t) = E(t, t)$; examples of varieties with pure MHS: smooth projective varieties, $\mathcal{M}_{\text{Dol}}$, $\mathcal{M}_{\text{DR}}$, Nakajima's quiver varieties

- in general the pure part of $H(M; x, y, t)$ is $PH(M; x, y) = \text{Coeff}_{T^0}\left(H(M; xT, yT, tT^{-1})\right)$; which, for a smooth M, is always the image of the cohomology of a smooth compactification

Connection to Arithmetic

Theorem 2 (...,Ito 2004, Katz 2005). *If M is a smooth quasi-projective variety defined over $\mathbb{Z}$ and*

$$\#\{M(\mathbb{F}_q)\} = E(q)$$

is a polynomial in q, then

$$E(M; x, y) = E(xy).$$

Example

$$\#\{\mathbb{P}^n(\mathbb{F}_q)\} = \#\{\mathbb{P}^{n-1}(\mathbb{F}_q)\} + \#\{\mathbb{A}^n(\mathbb{F}_q)\} = q^n + q^{n-1} + \cdots + q + 1$$

$$\Downarrow$$

$$E(\mathbb{P}^n, x, y) = (xy)^n + (xy)^{n-1} + \cdots + xy + 1$$

$$\Downarrow$$

$$P(\mathbb{P}^n, t) = t^{2n} + t^{2n-2} + \cdots + t^2 + 1$$

Fourier Transform for $T^*\mathbb{C}P^n$

- Calabi's hyperkähler manifold: $T^*\mathbb{C}P^n \cong \{(v,w) \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} | \sum_{i=1}^{n+1} v_i w_i = 1\}//GL(1)$

- $f(\xi) = q\delta_0 + (q-1)1 =$

$$\#\{(v,w) \in \mathbb{F}_q \times \mathbb{F}_q | vw = \xi\} = \begin{cases} 2q-1 & \text{if } \xi = 0 \\ q-1 & \text{if } \xi \neq 0 \end{cases}$$

- $\frac{1}{q-1}\#\{(v,w) \in \mathbb{F}_q^{n+1} \times \mathbb{F}_q^{n+1} | \sum_i^n v_i w_i = 1\} =$

$$\frac{1}{q-1} f \star f \star \cdots \star f(1) =$$

$$\frac{q^{n/2}}{q-1} \sum_{X \in \mathbb{F}_q} \mathcal{F}(f)(X)^{n+1} \Psi(X) = \frac{q^{n/2}}{q-1}$$

$$\sum_{X \in \mathbb{F}_q} \left(q q^{-1/2} 1(X) + (q-1) q^{1/2} \delta_0(X) \right)^{n+1} \Psi(X)$$

$$= \frac{q^{2n+1} - q^n}{q-1} = q^n (q^n + q^{n-1} + \cdots + 1)$$

- $\Rightarrow P(T^*\mathbb{C}P^n; t) = 1 + t^2 + t^4 + \cdots + t^{2n}$

Fourier Transform for $\mathcal{M}_{\mathsf{B}}$

Setup:

- $G = GL(n)$
- $C = \mathbb{P}^1$, with punctures $a_1, \ldots, a_k \in \mathbb{P}^1$
- $\tilde{\mathcal{C}}_i \subset GL(n)$ fixed semisimple conjugacy classes
- $\mathcal{M}_{\mathsf{B}} =$
  $\{(A_1, A_2, \ldots, A_k)\, A_i \in \tilde{\mathcal{C}}_i | A_1 \cdot \cdots \cdot A_k = I\}//G(\mathbb{C})$

Theorem 3 (Hausel–R-Villegas 2004).

$$\#\{\mathcal{M}_{\mathsf{B}}(\mathbb{F}_q)\} = \sum_{\chi \in Irr(G(\mathbb{F}_q))} \frac{\chi(1)^2 |Z(G(\mathbb{F}_q))|}{|G(\mathbb{F}_q)|^2} \prod_i \frac{\chi(\tilde{\mathcal{C}}_i(\mathbb{F}_q))}{\chi(1)} |\tilde{\mathcal{C}}_i(\mathbb{F}_q)|$$

is a polynomial $E(q)$ *in* $q \Rightarrow$

$$E(\mathcal{M}_{\mathsf{B}}, x, y) = E(xy)$$

.

Example Assume $n = 3$, and all the conjugacy classes $\tilde{C}_i$ are regular semisimple:

$$E(\mathcal{M}_{\mathsf{B}};q) =$$
$$\frac{\left(\left(q+1\right)\left(q^2+q+1\right)\right)^k}{\left(q^3-1\right)^2\left(q^2-1\right)^2} - \frac{\left(3\,q^2\left(q+1\right)\right)^k}{q^4\left(q^2-1\right)^2\left(q-1\right)^2}$$
$$+ 1/3\,\frac{\left(6\,q^3\right)^k}{q^6\left(q-1\right)^4} + \frac{\left(2\,q^2\left(q^2+q+1\right)\right)^k}{q^4\left(q^3-1\right)^2\left(q-1\right)^2}$$
$$+\frac{\left(q^3\left(q+1\right)\left(q^2+q+1\right)\right)^k}{q^6\left(q^3-1\right)^2\left(q^2-1\right)^2} - \frac{\left(3\,q^3\left(q+1\right)\right)^k}{q^6\left(q^2-1\right)^2\left(q-1\right)^2}.$$

Conjecture 3 (Hausel 2004). *When* $n=3$, $\tilde{\mathcal{C}}_i$ *are regular semisimple,* $h^{i-j}_{N-j}=h^{i+j}_{N+j}$ *for*

$$H(\mathcal{M}_{\mathsf{B}},q,t)=\sum h^i_j q^j t^i =$$

$$\frac{\left(\left(qt^2+1\right)\left(q^2t^4+qt^2+1\right)\right)^k}{\left(q^3t^6-1\right)\left(q^3t^4-1\right)\left(q^2t^4-1\right)\left(q^2t^2-1\right)}$$

$$-\frac{\left(3\,q^2t^4\left(qt^2+1\right)\right)^k}{q^4t^8\left(q^2t^4-1\right)\left(q^2t^2-1\right)\left(qt^2-1\right)(q-1)}$$

$$+1/3\,\frac{6^k\left(qt^2\right)^{3\,k}}{q^6t^{12}\left(qt^2-1\right)^2(q-1)^2}$$

$$+\frac{\left(q^2t^4\left(2\,q^2t^2+qt^2+q+2\right)\right)^k}{q^4t^8\left(q^3t^4-1\right)\left(q^3t^2-1\right)\left(qt^2-1\right)(q-1)}$$

$$+\frac{\left(q^3t^6(q+1)\left(q^2+q+1\right)\right)^k}{q^6t^{12}\left(q^3t^2-1\right)\left(q^3-1\right)\left(q^2t^2-1\right)\left(q^2-1\right)}$$

$$-\frac{\left(3\,q^3t^6(q+1)\right)^k}{q^6t^{12}\left(q^2t^2-1\right)\left(q^2-1\right)\left(qt^2-1\right)(q-1)},$$

Conjecture 4 (Hausel 2005). *If $\mathcal{M}_\mathsf{B}$ is the $GL(n,\mathbb{C})$ character variety of $\mathbb{P}^1$ punctured with k generic regular semisimple conjugacy classes:*

$$H(\mathcal{M}_\mathsf{B};q,t) = \sum_{\substack{\lambda^1,\lambda^2,\ldots,\lambda^l \\ |\lambda^1|+\cdots+|\lambda^l|=n}} A(\lambda_1,\ldots,\lambda_l),$$

where

$$A(\lambda_1,\ldots,\lambda_l) = \frac{(r)!\left(n!(qt^2)^{\frac{n(n-1)}{2}}\prod_{i=1}^{l}\frac{1}{|\lambda^i|!}X^{\lambda^i}_{(1^n)}((qt^2)^{-1},q)\right)^k}{(-1)^{r-1}r_1!r_2!\ldots r_s!(qt^2)^{n(\lambda^1,\ldots,\lambda^l)}\prod_{i=1}^{l}c_{\lambda^i}(q,t)c'_{\lambda^i}(q,t)}.$$

Here X^λ_ρ is Macdonald's (q,t) Green polynomial, which is defined as

$$X^\lambda_\rho(u,v) = \sum_{|\mu|=n} K_{\lambda\mu}(u,v)K_{\mu\rho}.$$

The Riemann-Hilbert map

- $\mathfrak{g} = \mathfrak{gl}(n)$
- $C = \mathbb{P}^1$ with punctures $a_1, \ldots, a_k \in \mathbb{P}^n$
- $\mathcal{C}_i$ semisimple adjoint orbit in $\mathfrak{g}(\mathbb{C})$
- $Q =$ $\{(A_1, \ldots, A_k)\, A_i \in \mathcal{C}_i | A_1 + \cdots + A_k = 0\} // G(\mathbb{C})$, Nakajima's star-saped quiver variety
- Q is smooth when $\mathcal{C}_i$ are generic
- "$Q \subset \mathcal{M}_{\mathsf{DR}}$", a point in Q gives the meromorphic flat $GL(n, \mathbb{C})$-connection $\sum A_i \frac{dz}{z - a_i}$ on the trivial bundle on C.
- $\tilde{\mathcal{C}}_i = \exp(2\pi i \mathcal{C}_i) \subset G(\mathbb{C})$ is the corresponding conjugacy class
- the *Riemann-Hilbert monodromy map*

$$\nu_a : Q \to \mathcal{M}_{\mathsf{B}}$$

  is given by sending the flat connection to its holonomy.

The purity conjecture

Conjecture 5. *If $\mathcal{C}_i$ are generic, then the Riemann-Hilbert map ν_a preserves mixed Hodge structures. Moreover it induces an isomorphism on the pure parts. As the mixed Hodge structure of Q is known to be pure, it follows, that the pure part of the cohomology of $\mathcal{M}_{\mathrm{B}}$ is isomorphic with the full cohomology of Q.*

Example

- $n = 3$, $k = 3$, $\mathcal{C}_i$ regular semisimple
- Q is $\cong$ to an E_6 ALE space,
- $\mathcal{M}_{\mathrm{B}} \cong \mathcal{M}_{\mathrm{Dol}}$ an elliptic fibration with singular fibre of type $\widehat{E}_6$.
- $P_t(Q) = 1 + 6t^2$
- $H(\mathcal{M}_{\mathrm{B}}; q, t) = 1 + 6qt^2 + q^2t^2$

$$\Downarrow$$

Conjecture is true in this case

Fourier transform for Q

- $\Psi : \mathbb{F}_q \to \mathbb{C}$ non-trivial additive character
- $f : \mathfrak{g}(\mathbb{F}_q) \to \mathbb{C}$ its Fourier transform $\mathcal{F}(f) : \mathfrak{g}^*(\mathbb{F}_q) \to \mathbb{C}$ at a $Y \in \mathfrak{g}^*(\mathbb{F}_q)$

  $$\mathcal{F}(f)(Y) := |\mathfrak{g}(\mathbb{F}_q)|^{-1/2} \sum_{X \in \mathfrak{g}(\mathbb{F}_q)} f(X)\Psi(\langle X, Y \rangle).$$

- $\delta_{\mathcal{C}_i} : \mathfrak{g}(\mathbb{F}_q) \to \mathbb{C}$ characteristic function of $\mathcal{C}_i \subset \mathfrak{g}(\mathbb{F}_q)$

Theorem 4 (Hausel–R-Villegas 2004).

$$\#\{Q(\mathbb{F}_q)\} = \frac{|Z(G(\mathbb{F}_q))||\mathfrak{g}(\mathbb{F}_q)|^{\frac{k-2}{2}}}{|G(\mathbb{F}_q)|} \sum_{X \in \mathfrak{g}(\mathbb{F}_q)} \mathcal{F}(\delta_{\mathcal{C}_1})(X) \cdot \cdots \cdot \mathcal{F}(\delta_{\mathcal{C}_k})(X)$$

Theorem 5 (Hausel–Letellier 2005). *When all $\mathcal{C}_i$ are generic regular semisimple then the pure part of the conjectured $H(\mathcal{M}_{\mathsf{B}}; q, t)$ polynomial agrees with the actual $P(Q; t)$.*

Example When $n = 3$:

$$P(Q;t) = \frac{\left(\left(t^2+1\right)\left(t^4+t^2+1\right)\right)^k}{\left(t^6-1\right)\left(t^4-1\right)} - \frac{\left(3\,t^4\left(t^2+1\right)\right)^k}{t^8\left(t^4-1\right)\left(t^2-1\right)} + 1/3\,\frac{6^k\left(t^2\right)^{3\,k}}{t^{12}\left(t^2-1\right)^2} - \frac{\left(t^4\left(t^2+2\right)\right)^k}{t^8\left(t^2-1\right)} + t^{6k-12} + \frac{(3\,t^6)^k}{t^{12}\left(t^2-1\right)}.$$

Purity conjecture for $g > 0$

Setup

- C genus g curve, punctures $a_1, \ldots, a_k \in C$
- $\tilde{\mathcal{C}}_i \subset G$ semisimple conjugacy classes
- $\mathcal{M}_\mathsf{B} = \{(A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_k)\, C_i \in \tilde{\mathcal{C}}_i |$
  $A_1^{-1} B_1^{-1} A_1 B_1 \ldots A_g^{-1} B_g^{-1} A_g B_g C_1 \ldots C_k = Id\} // G(\mathbb{C}$
- $\mathcal{C}_i \subset \mathfrak{g}$ semisimple adjoint orbit
- $Q = \{(A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_k)\, C_i \in \tilde{\mathcal{C}}_i |$
  $$A_1 B_1 - B_1 A_1 + \cdots + A_g B_g - B_g A_g + C_1 + \cdots + C_k = 0\} // G(\mathbb{C})$$

Conjecture 6. *The pure part of the cohomology of $\mathcal{M}_\mathsf{B}$ is isomorphic with the cohomology of Q, in particular: $PH(\mathcal{M}_\mathsf{B}; q, t) = P(Q; t)$*

Fourier transform for $(\mathbb{C}^2)^{[n]}$

- $g = 1$ and $k = 1$, and $\mathcal{C}_1 \subset GL(n)$ is a smallest non-central semisimple orbit
- $\mathcal{M}_{\mathrm{B}} =$

  $$\{(A_1, B_1, C)C \in \mathcal{C}_1 | A_1^{-1} B_1^{-1} A_1 B_1 C = Id\}//GL(n)$$

- $Q \cong (\mathbb{C}^2)^{[n]}$
- Conjecture $\Rightarrow PH^*(\mathcal{M}_{\mathrm{B}}) \cong H^*((\mathbb{C}^2)^{[n]}, \mathbb{C})$.
- **[Nevins-Stafford 2003]** $\Rightarrow$

  $$\mathcal{M}_{\mathrm{B}} \cong (\mathbb{C}^\times \times \mathbb{C}^\times)^{[n]}$$

Conjecture 7 (Hausel 2005). *In this case:*

$$\sum_{n=1}^{\infty} H(\mathcal{M}_{\mathrm{B}}; q, t) T^n = \prod_{m=1}^{\infty} \frac{(1 + q^m t^{2m-1} T^m)^2}{(1 - q^{m-1} t^{2m-2} T^m)(1 - q^{m+1} t^{2m} T^m)}$$

Questions: What about Mirror symmetry?

Conjecture 8 (Hausel 2004). $d, e \in \mathbb{Z}$, $(d,n) = (e,n) = 1$*:*

$$H_{\mathsf{st}}^{B^e}(x,y,t;\mathcal{M}_{\mathsf{B}}(SL(n,\mathbb{C}))) = H_{\mathsf{st}}^{\hat{B}^d}(x,y,t;\mathcal{M}_{\mathsf{B}}^e(PGL(n,\mathbb{C}))),$$

where H_{st}^B *is the stringy mixed Hodge polynomial twisted with a* B*-field, which can be defined identically as* E_{st}^B*.*

- Is there Arithmetic Fourier Transform method for E_{st}? And for E_{st}^B?
- Is there a larger representation theory picture, from which all the formulas would come naturally?
- Is there a Betti-version of the Geometric Langlands Program?
- Does understanding of the action of the mapping class group on our formulas lead to 3-manifold or knot invariants?