

# Arithmetic and physics of Higgs moduli spaces III

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Geometry and physics  
Montreal, June 2013

# Diffeomorphic spaces in non-Abelian Hodge theory

- $C$  genus  $g$  curve; fix group  $GL_n$

$$\mathcal{M}_{\text{Dol}}^d := \left\{ \begin{array}{l} \text{moduli space of semistable rank } n \\ \text{degree } d \text{ Higgs bundles } (E, \phi) \\ \phi \in H^0(C, \text{End}(E) \otimes K) \text{ Higgs field} \end{array} \right\}$$

$$\mathcal{M}_{\text{B}}^d := \{A_1, B_1, \dots, A_g, B_g \in GL_n \mid \prod_{i=1}^g A_i^{-1} B_i^{-1} A_i B_i = e^{\frac{2\pi i d}{n}} Id\} // PGL_n$$

when  $(d, n) = 1$  these are smooth non-compact varieties

- Non-Abelian Hodge Theorem:  $\mathcal{M}_{\text{Dol}}^d \stackrel{\text{diff}}{\cong} \mathcal{M}_{\text{B}}^d$   
(Hitchin, Donaldson, Corlette, Simpson)
- $g = 1 \rightsquigarrow$  Stone-von Neumann  $\rightsquigarrow$   
 $\mathcal{M}_{\text{B}}^d \cong (\mathbb{C}^*)^2 \cong T^*\text{Jac}(C) \cong \mathcal{M}_{\text{Dol}}^d$
- Problem: what is Poincaré polynomial  
 $P(\mathcal{M}_{\text{Dol}}^d; t) = P(\mathcal{M}_{\text{B}}^d; t)?$

# Conjectured mixed Hodge polynomials

Conjecture (Hausel-Villegas, 2008)

$$\sum_{\lambda} \prod \frac{(z^{2l+1} - w^{2a+1})^{2g}}{(z^{2l+2} - w^{2a})(z^{2l} - w^{2a+2})} T^{|\lambda|} = \exp \left( \sum_{n,k} \frac{H(\mathcal{M}_{\mathbb{B}}^d; w^{2k}, -(zw)^{-2k})(zw)^{dn}}{(z^{2k}-1)(1-w^{2k})} \frac{T^{nk}}{k} \right)$$

- when  $g = 1$   $\mathcal{M}_{\mathbb{B}}^d = (\mathbb{C}^*)^2$  by Stone-von Neumann  $\overset{HV}{\rightsquigarrow}$

$$\sum_{\lambda} \prod \frac{(z^{2l+1} - w^{2a+1})^2}{(z^{2l+2} - w^{2a})(z^{2l} - w^{2a+2})} T^{|\lambda|} = \exp \left( \sum_{k \geq 1} \frac{(z^k - w^k)^2}{(z^{2k}-1)(1-w^{2k})(1-T^k)} \frac{T^k}{k} \right)$$

# Curious Hard Lefschetz

- (Hausel-Villegas 2008) calculates

$$E(\mathcal{M}_B^d; q) = |\mathcal{M}_B^d(\mathbb{F}_q)| = \sum_{\chi \in \text{Irr}(\text{GL}_n(\mathbb{F}_q))} \frac{|\text{GL}_n(\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-2}} \frac{\chi(\xi_n^d)}{\chi(1)}$$

- we find  $E(\mathcal{M}_B^d; q) = q^{d_n} E(\mathcal{M}_B^d; 1/q)$  palindromic  
by *Alvis-Curtis duality*

$$q^{\frac{n(n-1)}{2}} \chi(1)(1/q) = \chi'(1)(q) \text{ for dual pair } \chi, \chi' \in \text{Irr}(\text{GL}_n(\mathbb{F}_q))$$

- $\rightsquigarrow$  Curious Hard Lefschetz Conjecture (theorem when  $n = 2$ ):

$$L^l : \underset{X}{\text{Gr}_{d_n-2l}^W(H^{i-l}(\mathcal{M}_B^d))} \rightarrow \underset{X \cup \alpha^l}{\text{Gr}_{d_n+2l}^W(H^{i+l}(\mathcal{M}_B^d))},$$

where  $\alpha \in W_4 H^2(\mathcal{M}_B^d)$

- The implied functional equation on the conjectured  $H(\mathcal{M}_B^d; q, t) = (qt)^{d_n} H(\mathcal{M}_B^d; \frac{1}{qt^2}, t)$  holds

# Perverse filtration

- $f : X \rightarrow Y$  a *proper* map between complex algebraic varieties of relative dimension  $d$
- (de Cataldo-Migliorini 2005) introduce *perverse filtration*  $P_0 \subset \dots \subset P_i \subset \dots \subset P_k(X) \cong H^k(X)$  from the study of the Beilinson-Bernstein-Deligne-Gabber decomposition theorem for  $Rf_*(\mathbb{Q}_X)$  into perverse sheaves
- recipe (de Cataldo-Migliorini, 2008) for perverse filtration when  $X$  smooth and  $Y$  affine:  
take  $Y_0 \subset \dots \subset Y_i \subset \dots \subset Y_d = Y$   
s.t.  $Y_i$  generic with  $\dim(Y_i) = i$  then

$$P_{k-i-1}H^k(X) = \ker(H^k(X) \rightarrow H^k(f^{-1}(Y_i)))$$

- the Relative Hard Lefschetz Theorem holds:

$$L^l : \begin{array}{ccc} Gr_{d-l}^P(H^*(X)) & \rightarrow & Gr_{d+l}^P H^{*+2l}(X) \\ x & \mapsto & x \cup \alpha^l \end{array}$$

where  $\alpha \in H^2(X)$  is a relative ample class

# Main conjecture

- recall Hitchin map  $\chi : \mathcal{M}_{\text{Dol}}^d \rightarrow \mathbb{A} := \bigoplus_{i=1}^n H^0(C; K^i)$   
 $(E, \phi) \mapsto \text{charpol}(\phi)$
- (Hitchin 1987)  $\rightarrow$  completely integrable Hamiltonian system and *proper*

## Conjecture ("P=W", de Cataldo-Hausel-Migliorini 2008)

$P_k(\mathcal{M}_{\text{Dol}}^d) \cong W_{2k}(\mathcal{M}_{\text{B}}^d)$  under the isomorphism

$H^*(\mathcal{M}_{\text{Dol}}^d) \cong H^*(\mathcal{M}_{\text{B}}^d)$  from non-Abelian Hodge theory.

In particular  $\text{CHL} \Leftrightarrow \text{RHL}$

## Theorem (de Cataldo-Hausel-Migliorini 2010)

$P = W$  for  $n = 2$ .

- proof mirroring Ngô's proof of the fundamental lemma
- evidence for  $n > 2$ ?

# Refined Gopakumar-Vafa conjecture for local curves

- we follow (Chuang-Diaconescu-Pan 2012)
- $Y$  total space of  $\mathcal{O}_C \oplus K_C$  over  $C$
- $Y$  CY 3-fold , "local curve"
- conjectural "quantum" Pandharipande-Thomas invariants

$$Z_{PT}^{ref} := \sum_{\beta \in H_2(Y)} \sum_{e \in \mathbb{Z}} T^\beta q^e E_{virt}(\mathcal{P}(Y, \beta, e); y)$$

- Gopakumar-Vafa generating function of refined BPS invariants:

$$F_{GV}^{ref} := \sum_{\substack{k \geq 1 \\ \beta \in H_2(Y)}} \sum_{j_L, l \geq 0} \frac{T^{k\beta}}{k} (-1)^{j_L+l} N_\beta^{(j_L, l)} \frac{(q^{-kj_L} + \dots + q^{kj_L}) q^{-k} y^l}{(1-(qy)^{-k})(1-(q/y)^{-k})}$$

Conjecture ("refined BPS", Chuang-Diaconescu-Pan 2012)

$$Z_{PT}^{ref} = \exp(F_{GV}^{ref})$$

- Gopakumar-Vafa's BPS invariants  $N_{\beta}^{j_L, j_R}$  heuristically arise from decomposing the cohomology  $H^*(\mathcal{M}_{\beta}^e)$  of the space of D-branes via a putative action of  $(\mathfrak{sl}_2)_L \times (\mathfrak{sl}_2)_R$
- for local curve  $Y$ , (Chuang-Diaconescu-Pan 2012) argue that  $\mathcal{M}_{\beta}^e \cong \mathcal{M}_{\text{Dol}}^d$  where  $\beta = n[C]$  and  $d = e + n(g - 1)$
- recall  $\chi : \mathcal{M}_{\text{Dol}}^d \rightarrow \mathbb{A}^1$  induces perverse filtration  $P$  on  $H^*(\mathcal{M}_{\text{Dol}}^d)$  with RHL
- RHL on  $\text{Gr}^P(H^*(\mathcal{M}_{\text{Dol}}^d)) \rightsquigarrow (\mathfrak{sl}_2)_L$  action on  $\text{Gr}^P(H^*(\mathcal{M}_{\text{Dol}}^d))$
- the corresponding primitive decomposition  $H^m(\mathcal{M}_{\text{Dol}}^d) \cong \bigoplus_{i,j} Q^{i,j;m}$  gives at least a  $(\mathfrak{gl}_1)_R$  action
- Chuang-Diaconescu-Pan define  $N_{\beta}^{j_L; l} := \dim(Q^{j_L, 0; l})$



# $Z_{PT}^{ref}$ via geometric engineering

- geometric engineering  $\leadsto Z_{PT}^{ref} = Z_{gauge}$   
 $Z_{gauge}$  partition function of certain gauge theory
- Chuang-Diaconescu-Pan argue that it should be a  $U(1)$ -gauge theory on  $X = \mathbb{R}^4 \cong \mathbb{C}^2$

- $Z_{gauge} = \sum_{k \geq 0} Q^k \chi_{Y'}^{\mathbb{T}^2} \left( \det(\mathcal{V}_k)^{1-g} \otimes (T^*X^{[k]})^{\oplus g} \right)$

$\mathcal{V}_k$  is the tautological bundle on the Hilbert scheme  $X^{[k]}$

$\mathbb{T}^2$  acts on  $X$  and so on  $X^{[k]}$  with isolated fixed points

$\leadsto Z_{gauge}$  is defined by localizing to the fixed points

- after changes of variables we have

$$\sum_{\lambda} \prod \frac{(z^{2l+1} - w^{2a+1})^{2g}}{(z^{2l+2} - w^{2a})(z^{2l} - w^{2a+2})} T^{|\lambda|} \stackrel{g.e.}{=} Z_{PT}^{ref} \stackrel{CDP}{=} \exp(F_{GV}) \Big|_{P=W}$$

$$\stackrel{HV}{=} \exp \left( \sum_{n,k} \frac{H(\mathcal{M}_B^d; w^{2k}, -(zw)^{-2k})(zw)^{dn}}{(z^{2k}-1)(1-w^{2k})} \frac{T^{nk}}{k} \right)$$

- conclusion:  $HV, CDP + g.e. \Rightarrow P = W$

- when  $g = 1$   $\mathcal{M}_B^d = (\mathbb{C}^\times)^2$  by Stone-von Neumann  $\overset{HV}{\rightsquigarrow}$

$$\sum_{\lambda} \prod \frac{(z^{2l+1} - w^{2a+1})^2}{(z^{2l+2} - w^{2a})(z^{2l} - w^{2a+2})} T^{|\lambda|} = \exp \left( \sum_{k \geq 1} \frac{(z^k - w^k)^2}{(z^{2k} - 1)(1 - w^{2k})(1 - T^k)} \frac{T^k}{k} \right)$$

- after geometric engineering this formula becomes

$$\sum_n \chi_y^{\mathbb{T}^2}(X^{[n]}) T^n = \sum_n \chi_{y, st}^{\mathbb{T}^2}(X^n / S_n) T^n$$

(Waelder, 2008) proves geometrically a more general DMVV formula for equivariant elliptic genus  $\rightsquigarrow$  our  $g = 1$  formula follows!

- (Chuang-Diaconescu-Pan 2012) refined BPS conjecture  $\Leftrightarrow$  (Hausel-Villegas 2008) conjecture on  $H(\mathcal{M}_B^d; q, t)$  provided  $P = W$  of (de Cataldo-Hausel-Migliorini 2010)
- studying wall-crossing for the stability condition for  $Z_{PT}^{ref}(Y) \rightsquigarrow$  recursive formulas (Chuang-Diaconescu-Pan 2010) for  $P_t(\mathcal{M}_{Dol}^d)$  studied by (Mozgovoy 2011)
- How does it reduce to (Pandharipande-Thomas 2010) when specializing to the unrefined BPS invariants?
- mathematically geometric engineering proposes a deep connection between  $K_{\mathbb{T}^2}((\mathbb{C}^2)^{[n]})$  and  $H^*(\mathcal{M}_{Dol}^d)$
- may lead to connections between (Haiman 2002) and (Hausel-Letellier-Villegas 2008) explaining the appearance of Macdonald polynomials in both
- DAHA acts on  $K_{\mathbb{T}^2}((\mathbb{C}^2)^{[n]})$  by (Gordon-Stafford 2004) it is expected that DAHA acts on  $H^*(\mathcal{M}_{Dol}^d)$  from (Yun 2009)
- are these DAHA actions related by geometric engineering?