

# Arithmetic and physics of Higgs moduli spaces II

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- $C$  genus  $g$  curve; fix  $r > 0$

$$\mathcal{M}_{\text{Dol}}^{d,r} := \left\{ \begin{array}{l} \text{moduli space of semistable rank } r \\ \text{degree } d \text{ Higgs bundles } (E, \phi) \\ \phi \in H^0(C, \text{End}(E) \otimes K) \text{ Higgs field} \end{array} \right\}$$

$$\mathcal{M}_{\text{B}}^{d,r} := \{A_1, B_1, \dots, A_g, B_g \in \text{GL}_r \mid \prod_{i=1}^g A_i^{-1} B_i^{-1} A_i B_i = e^{\frac{2\pi i d}{r}} \text{Id}\} // \text{PGL}_r$$

when  $(d, r) = 1$  these are smooth non-compact varieties

- Non-Abelian Hodge Theorem:  $\mathcal{M}_{\text{Dol}}^{d,r} \stackrel{\text{diff}}{\cong} \mathcal{M}_{\text{B}}^{d,r}$   
[Hitchin, Donaldson, Corlette, Simpson]
- Problem:  $P(\mathcal{M}_{\text{Dol}}^{d,r}; t) = P(\mathcal{M}_{\text{B}}^{d,r}; t)$ ?

- when  $r = 1 \rightsquigarrow$

$$\phi : H^0(C; \text{End}(E) \otimes K_C) \cong H^0(C; K_C) \rightsquigarrow$$

$$\mathcal{M}_{\text{Dol}}^{d,1} \cong T^* J^d(C) \cong J^d(C) \times H^0(C; K_C)$$

$\cong_{\text{diff}}$

$$\mathcal{M}_{\text{B}}^{d,1} \cong \text{Hom}(\pi_1(C), \text{GL}_1(\mathbb{C})) \cong (\mathbb{C}^\times)^{2g}$$

- $r > 1$  and  $g = 0$  both  $\mathcal{M}_{\text{B}}^{d,r}$  and  $\mathcal{M}_{\text{Dol}}^{d,r}$  are empty
- $r > 1$  and  $g = 1 \rightsquigarrow$  Stone-von Neumann  $\rightsquigarrow$

$$\mathcal{M}_{\text{B}}^{d,r} \cong (\mathbb{C}^*)^2$$

$\cong_{\text{diff}}$

$$\mathcal{M}_{\text{Dol}}^{d,r} \cong T^* J^d(C) \cong T^* C \cong C \times H^0(C; K_C) \cong C \times \mathbb{C}$$

# Mixed Hodge polynomials

- (Deligne 1971) proved the existence of  $W_0 \subset \cdots \subset W_i \subset \cdots \subset W_{2k} = H^k(X; \mathbb{Q})$  for any complex algebraic variety  $X$  which is
  - functorial
  - compatible with cup-product
- $H(X; q, t) = \sum \dim(W_i/W_{i-1}(H^k(X))) t^k q^{\frac{i}{2}}$ , *mixed Hodge polynomial*
- $P(X; t) = H(X; 1, t)$ , *Poincaré polynomial*
- $E(X; q) = q^D H(X; 1/q, -1)$ , *E-polynomial of  $X$ .*

## Theorem (Katz 2008)

If  $M$  is a smooth quasi-projective variety defined over  $\mathbb{Z}$  and

$$\#\{M(\mathbb{F}_q)\} = E(q)$$

is a polynomial in  $q$ , then  $E(M; q) = E(q)$ .

# Examples for mixed Hodge polynomials

- $M$  smooth semi-projective variety  $\rightsquigarrow$   
 $W_{k-1}(H^k(M)) = 0$  because it is smooth  
 $W_k(H^k(M)) = W_k(H^k(C)) = H^k(C) = H^k(M) \leftarrow C$  projective  
such cohomology is called *pure*  
 $\rightsquigarrow H(M; q, t) = P(M; tq^{\frac{1}{2}})$   
 $\rightsquigarrow E(M; q) = q^D P(M; -q^{-\frac{1}{2}})$
- $H(T^*J^d(C); q, t) = H(J^d(C); q, t) = (1 + q^{\frac{1}{2}}t)^{2g}$  MHS is pure  
 $E(T^*J^d(C); q) = q^g(1 - q^{\frac{1}{2}})^{2g}$  not polynomial count
- $\#(\mathrm{GL}_1(\mathbb{F}_q)) = q - 1 \stackrel{\text{Katz}}{\Rightarrow} E(\mathrm{GL}_1(\mathbb{C}); q) = E(\mathbb{C}^\times; q) = q - 1 \rightsquigarrow$   
 $W_1 H^1(\mathbb{C}^\times) = 0 \rightsquigarrow H(\mathbb{C}^\times; q, t) = 1 + qt$  MHS not pure
- $\rightsquigarrow H((\mathbb{C}^\times)^{2g}; q, t) = (1 + qt)^{2g}$  and  $E((\mathbb{C}^\times)^{2g}; q) = (q - 1)^{2g}$
- $E((\mathbb{C}^\times)^{2g}; q)$  palindromic  $\Leftrightarrow E((\mathbb{C}^\times)^{2g}; q) = q^{2g} E((\mathbb{C}^\times)^{2g}; 1/q)$
- extends to  $H(\mathbb{C}^\times)^{2g}; q, t) = (qt)^{2g} H(\mathbb{C}^\times)^{2g}; 1/qt^2, t)$

Theorem (Frobenius 1896, Hurwitz 1902, Freed-Quinn 1993,...)

Let  $z \in G$  in a finite group  $G$  then

$$\begin{aligned} \#\{a_j, b_j \in G \mid [a_1, b_1] \dots [a_g, b_g] = z\} &= \\ &= \sum_{\chi \in \text{Irr}(G)} \frac{|G|^{2g-1} \chi(z)}{\chi(1)^{2g-2} \chi(1)} \end{aligned}$$

- (Hausel-Villegas 2008) calculates

$$E(\mathcal{M}_B^{d,r}; q) \stackrel{\text{Katz}}{=} |\mathcal{M}_B^{d,r}(\mathbb{F}_q)| = \sum_{\chi \in \text{Irr}(\text{GL}_r(\mathbb{F}_q))} \frac{|\text{GL}_r(\mathbb{F}_q)|^{2g-2} (q-1) \chi(\xi_r^d)}{\chi(1)^{2g-2} \chi(1)}$$

- $\leadsto E(\mathcal{M}_B^{d,r}; q) = E(\mathcal{M}_B^{d',r}; q)$  when  $(d, r) = (d', r) = 1$
- $\mathcal{M}_B^{d,r}$  and  $\mathcal{M}_B^{d',r}$  Galois conjugate  $\Rightarrow$   
 $H(\mathcal{M}_B^{d,r}; q, t) = H(\mathcal{M}_B^{d',r}; q, t)$

# Character table of $\mathrm{GL}_2(\mathbb{F}_q)$

Table 1: characters of  $\mathbf{GL}_2(\mathbb{F}_q)$   
 (note that  $|\mathbf{GL}_2(\mathbb{F}_q)| = q(q-1)^2(q+1)$ )

Classes	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ $a \in \mathbb{F}_q^\times$	$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ $a, b \in \mathbb{F}_q^\times$ $a \neq b$	$\begin{pmatrix} x & 0 \\ 0 & {}^F x \end{pmatrix}$ $x \in \mathbb{F}_{q^2}^\times$ $x \neq {}^F x$	$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ $a \in \mathbb{F}_q^\times$
Number of classes of this type	$q-1$	$\frac{(q-1)(q-2)}{2}$	$\frac{q(q-1)}{2}$	$q-1$
Cardinal of the class	1	$q(q+1)$	$q(q-1)$	$q^2-1$
$R_{\mathbf{T}}^G(\alpha, \beta)$ $\alpha, \beta \in \mathrm{Irr}(\mathbb{F}_q^\times)$ $\alpha \neq \beta$	$(q+1)\alpha(a)\beta(a)$	$\alpha(a)\beta(b) + \alpha(b)\beta(a)$	0	$\alpha(a)\beta(a)$
$-R_{\mathbf{T}_s}^G(\omega)$ $\omega \in \mathrm{Irr}(\mathbb{F}_{q^2}^\times)$ $\omega \neq \omega^q$	$(q-1)\omega(a)$	0	$-\omega(x) - \omega({}^F x)$	$-\omega(a)$
$\mathrm{Id}_{\mathbf{G}}.(\alpha \circ \det)$ $\alpha \in \mathrm{Irr}(\mathbb{F}_q^\times)$	$\alpha(a^2)$	$\alpha(ab)$	$\alpha(x \cdot {}^F x)$	$\alpha(a^2)$
$\mathrm{St}_{\mathbf{G}}.(\alpha \circ \det)$ $\alpha \in \mathrm{Irr}(\mathbb{F}_q^\times)$	$q\alpha(a^2)$	$\alpha(ab)$	$-\alpha(x \cdot {}^F x)$	0

# Example $GL_2(\mathbb{F}_q)$



$$\frac{1}{|\mathrm{PGL}_2(\mathbb{F}_q)|(q-1)^{2g}} \#\{a_j, b_j \in \mathrm{GL}_2(\mathbb{F}_q) \mid [a_1, b_1] \dots [a_g, b_g] = -1\} =$$

$$\frac{1}{|\mathrm{PGL}_2(\mathbb{F}_q)|(q-1)^{2g}} \sum_{\chi \in \mathrm{Irr}(\mathrm{GL}_2(\mathbb{F}_q))} \frac{|\mathrm{GL}_2(\mathbb{F}_q)|^{2g-1} \chi(-1)}{\chi(1)^{2g-2} \chi(1)} =$$

$$(q^2-1)^{2g-2} + q^{2g-2}(q^2-1)^{2g-2} - \frac{1}{2}q^{2g-2}(q-1)^{2g-2} - \frac{1}{2}q^{2g-2}(q+1)^{2g-2}$$

$$\stackrel{\text{Katz}}{=} \frac{E(\mathcal{M}_B^{d,2})}{(q-1)^{2g}}.$$

- e.g.  $g = 0$  gives 0 when  $g = 1$  it gives 1



## Conjecture (Hausel-Villegas, 2008)

$$\sum_{\lambda} \prod \frac{(z^{2l+1}-w^{2a+1})^{2g}}{(z^{2l+2}-w^{2a})(z^{2l}-w^{2a+2})} T^{|\lambda|} = \exp \left( \sum_{r,k} \frac{H(\mathcal{M}_B^{d,r}; w^{2k}, -(zw)^{-2k})(zw)^{Dr}}{(z^{2k}-1)(1-w^{2k})} \frac{T^{rk}}{k} \right)$$

- when  $g = 0$   $\mathcal{M}_B^{d,r}$  empty unless  $r = 1$  when  $\mathcal{M}_B^{d,1} = pt \stackrel{HV}{\sim}$

$$\sum_{\lambda} \prod \frac{T^{|\lambda|}}{(z^{2l+2}-w^{2a})(z^{2l}-w^{2a+2})} = \exp \left( \sum_{k \geq 1} \frac{1}{(z^{2k}-1)(1-w^{2k})} \frac{T^k}{k} \right)$$

proved by [Garsia–Haiman, 1996]

- when  $g = 1$   $\mathcal{M}_B^{d,r} = (\mathbb{C}^*)^2$  by Stone-von Neumann  $\stackrel{HV}{\sim}$

$$\sum_{\lambda} \prod \frac{(z^{2l+1}-w^{2a+1})^2}{(z^{2l+2}-w^{2a})(z^{2l}-w^{2a+2})} T^{|\lambda|} = \exp \left( \sum_{k \geq 1} \frac{(z^k-w^k)^2}{(z^{2k}-1)(1-w^{2k})(1-T^k)} \frac{T^k}{k} \right)$$

proved geometrically by [Waelder, 2008]

# Formula for $H(\mathcal{M}_B^{d,2}; q, t)$

- when  $r = 2$  Conjecture  $\leadsto$

$$\begin{aligned} \frac{H_2(q, t)}{(1 + qt)^{2g}} &= \frac{(q^2 t^3 + 1)^{2g}}{(q^2 t^2 - 1)(q^2 t^4 - 1)} + \frac{q^{2g-2} t^{4g-4} (q^2 t + 1)^{2g}}{(q^2 - 1)(q^2 t^2 - 1)} - \\ &- \frac{1}{2} \frac{q^{2g-2} t^{4g-4} (qt + 1)^{2g}}{(qt^2 - 1)(q - 1)} - \frac{1}{2} \frac{q^{2g-2} t^{4g-4} (qt - 1)^{2g}}{(q + 1)(qt^2 + 1)} \end{aligned}$$

- $H(\mathcal{M}_B^{d,2}; q, t) \stackrel{HV}{=} H_2(q, t)$
- $H_2(q, -1) = H_2(1/q, -1) q^{8g-6} = E(\mathcal{M}_B^{d,2}; q)$
- $\frac{H_2(1, t)}{(1+t)^{2g}}$  agrees with [Hitchin 1987] for  $P(\mathcal{M}_B^{d,2}; t)$

$$\begin{aligned} P_t(\mathcal{M}) &= \frac{(1 + t^3)^{2g}}{(1 - t^2)(1 - t^4)} - \frac{t^{4g-4} (1 - t)^{2g}}{4(1 + t^2)} \\ &- \frac{t^{4g-3} (1 + t)^{2g-2} (g - 1)}{(1 - t)} + \frac{t^{4g-4} (1 + t)^{2g-2} (t^2 - 4t + 1)}{4(1 - t)^2} \end{aligned}$$

- [Hausel–Villegas 2008]  $\rightsquigarrow H(\mathcal{M}_B^{d,2}; q, t) = H_2(q, t)$   
proved by finding the weights of universal generators and extending it to a monomial basis of [Hausel–Thaddeus 2002]
- [Hausel–Villegas 2008]  $\rightsquigarrow$  for  $r = 3$  conjecture is consistent with [Gothen, 1994] for  $P(\mathcal{M}_{\text{Dol}}^{d,3}; t)$
- [Garcia-Prada–Heinloth–Schmitt 2011]  $\rightsquigarrow$  for  $r = 4$  conjecture is consistent with their computation  $P_t(\mathcal{M}_{\text{Dol}}^{d,4})$
- [Garcia-Prada–Heinloth 2012]  $\rightsquigarrow$  prove Hirzebruch  $y$ -genus specialization of extended conjecture for every  $r$
- [Chaudouard–Laumon 2012]  $\rightsquigarrow$  conjecture is consistent with the nilpotent part of the Arthur-Selberg trace formula
- [Chuang–Diaconescu–Pan 2012]  $\rightsquigarrow$  conjecture is equivalent with refined Gopakumar-Vafa conjecture for local curve Calabi-Yau 3-fold provided  $P = W$

# Character table for $GL_3(\mathbb{F}_q)$

$$|GL_3(\mathbb{F}_q)| = (q^3-1)(q^3-q)(q^3-q^2)$$

$$|T^F| = (q-1)^3$$

$$|T_1^F| = (q^2-1)(q-1)$$

$$|T_2^F| = q^2-1$$

$$|W(T^F)| = 6$$

$$|W(T_1^F)| = 2$$

$$|W(T_2^F)| = 3$$

$$q^3(q-1)^3(q+1)(q^2+q+1)$$

Tableau des caractères de  $GL_3(\mathbb{F}_q)$

$$|C_6(\begin{smallmatrix} 1 & & \\ & 1 & \\ & & 1 \end{smallmatrix})^F| = q^3(q-1)^2$$

$$|C_6(\begin{smallmatrix} 1 & & \\ & 1 & \\ & & \omega \end{smallmatrix})^F| = q^2(q-1)$$

$$|C_6(\begin{smallmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{smallmatrix})^F| = (q^2-1)(q+1)$$

$$|C_6^F(\begin{smallmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{smallmatrix})| = (q^2-1)(q^2-q)$$

	$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$ $a \in \mathbb{F}_q^*$	$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}$ $a \neq b, a, b \in \mathbb{F}_q^*$	$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ $a, b, c \in \mathbb{F}_q^*, 2a \neq b$	$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$ $\lambda_1 \in \mathbb{F}_q^*, \lambda_2 \in \mathbb{F}_q^* - \mathbb{F}_q$	$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1^q \end{pmatrix}$ $\lambda_1 \in \mathbb{F}_q^* - \mathbb{F}_q^2$	$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}$ $a \in \mathbb{F}_q^*$	$\begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}$ $a \in \mathbb{F}_q^*$	$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 1 \\ 0 & 0 & b \end{pmatrix}$ $a \neq b \in \mathbb{F}_q^*$
Nombre de classes de ce type	$q-1$	$(q-1)(q-2)$	$\frac{(q-1)(q-2)(q-3)}{6}$	$\frac{q(q-1)}{6}$	$\frac{q(q^2-q)}{6}$	$q-1$	$q-1$	$(q-1)(q-2)$
Cardinal des classes	1	$q^2(q^2+q+1)$	$q^3(q+1)(q^2+q+1)$	$q^2(q^2+q+1)(q^2-1)$	$(q^2-q)(q^2-q^2)$	$(q^3-1)(q+1)$	$(q^3-1)(q^2-q)$	$q^2(q^2-1)(q+1)$
$R_T^F(\chi, \beta, \gamma)$ $\chi, \beta, \gamma \in \text{Irr}(\mathbb{F}_q^*)$ $(\chi, \beta, \gamma) \neq (1, 1, 1)$	$(q+1)(q^2+q+1)$ $\times \alpha(a)\beta(a)\gamma(a)$	$(q+1)\alpha(a)\beta(a)\gamma(a)$ $+ \alpha(b)\beta(a)\gamma(a)$ $+ \alpha(a)\beta(b)\gamma(a)$	$\sum_{\sigma \in S_3} \alpha(\sigma a)\beta(\sigma b)\gamma(\sigma c)$	0	0	$(1+2q)$ $\times \alpha(a)\beta(a)\gamma(a)$	$\alpha(a)\beta(a)\gamma(a)$	$\alpha(a)\beta(b)\gamma(b)$ $+ \alpha(b)\beta(a)\gamma(b)$ $+ \alpha(b)\beta(b)\gamma(a)$
Id $\mathbb{F}_q^*$ (x odet) $\alpha \in \text{Irr}(\mathbb{F}_q^*)$	$\alpha(a^3)$	$\alpha(a^2b)$	$\alpha(abc)$	$\alpha(\lambda_1^F \lambda_1^F \lambda_2)$	$\alpha(\lambda_1^F \lambda_1^F \lambda_1^q)$	$\alpha(a^3)$	$\alpha(a^3)$	$\alpha(ab^2)$
St $\mathbb{F}_q^*$ (x odet) $\alpha \in \text{Irr}(\mathbb{F}_q^*)$	$q^3 \alpha(a^3)$	$q \alpha(a^2b)$	$\alpha(abc)$	$-\alpha(\lambda_1^F \lambda_1^F \lambda_2)$	$\alpha(\lambda_1^F \lambda_1^F \lambda_1^q)$	0	0	0
$R_{T_1}^F(w, w \neq w^q)$ $w \in \text{Irr}(\mathbb{F}_q^*)$	$(q^2-1)(q-1)w(a)$	0	0	0	$w(\lambda_1) + w(\lambda_1^q)$ $+ w(\lambda_1^{q^2})$	$(q-1)w(a)$	$w(a)$	0
$-R_{T_1}^F(w, w)$ $w \in \text{Irr}(\mathbb{F}_q^*), w \neq 1$	$(q^2-1)w(a)\alpha(a)$	$(q-1)w(a)\alpha(b)$	0	$-w(\lambda_1)\alpha(\lambda_1)$ $-w(\lambda_1^q)\alpha(\lambda_1^q)$	0	$-w(a)\alpha(a)$	$-w(a)\alpha(a)$	$-w(b)\alpha(b)$
$R_{T_2}^F(\alpha \text{ odet})$ $\alpha \in \text{Irr}(\mathbb{F}_q^*)$	$q(q+1)\alpha(a^3)$	$(q+1)\alpha(a^2b)$	$2\alpha(abc)$	0	$-\alpha(\lambda_1^F \lambda_1^F \lambda_1^q)$	$q\alpha(a^3)$	0	$\alpha(ab^2)$
$R_{C_6(S)}^F(\text{Id}, \alpha \neq \beta \in \text{Irr}(\mathbb{F}_q^*))$	$(q^2+q+1)\alpha(a^3)\beta(a)$	$\alpha(a^2)\beta(b) + \alpha(a)\alpha(b)\beta(a)$	$\alpha(ab)\beta(c) + \alpha(ac)\beta(b) + \alpha(bc)\beta(a)$	$\alpha(\lambda_1^{q^2+1})\beta(\lambda_1)$	0	$(q+1)\alpha(a^3)\beta(a)$	$\alpha(a^2)\beta(a)$	$\alpha(ab)\beta(b)$ $+ \alpha(b^2)\beta(a)$
$R_{C_6^F(S)}^F(\text{St}, \alpha \neq \beta \in \text{Irr}(\mathbb{F}_q^*))$	$q(q^2+q+1)$ $\times \alpha(a^2)\beta(a)$	$q\alpha(a^2)\beta(b) + (q+1)\alpha(a)\alpha(b)\beta(a)$	$\alpha(ab)\beta(c) + \alpha(ac)\beta(b) + \alpha(bc)\beta(a)$	$-\alpha(\lambda_1^{q^2+1})\beta(\lambda_1)$	0	$q\alpha(a^2)\beta(a)$	0	$\alpha(b^2)\beta(a)$

# Our conjecture for $H(\mathcal{M}^{d,3}; q, t)$

$$\begin{aligned}
 & \frac{(q^3 t^5 - 1)^{2g} (q^2 t^3 - 1)^{2g}}{(q^3 t^6 - 1)(q^3 t^4 - 1)(q^2 t^4 - 1)(q^2 t^2 - 1)} + \frac{q^{6g-6} t^{12g-12} (q^3 t - 1)^{2g} (q^2 t - 1)^{2g}}{(q^3 t^2 - 1)(q^3 - 1)(q^2 t^2 - 1)(q^2 - 1)} + \\
 & + \frac{q^{4g-4} t^{8g-8} (q^3 t^3 - 1)^{2g} (qt - 1)^{2g}}{(q^3 t^4 - 1)(q^3 t^2 - 1)(qt^2 - 1)(q - 1)} + \frac{1}{3} \frac{q^{6g-6} t^{12g-12} ((qt - 1)^{2g})^2}{(qt^2 - 1)^2 (q - 1)^2} - \\
 & - \frac{1}{3} \frac{q^{6g-6} t^{12g-12} (q^2 t^2 + qt + 1)^{2g}}{(q^2 t^4 + qt^2 + 1)(q^2 + q + 1)} - \frac{q^{4g-4} t^{8g-8} (q^2 t^3 - 1)^{2g} (qt - 1)^{2g}}{(q^2 t^4 - 1)(q^2 t^2 - 1)(qt^2 - 1)(q - 1)} - \\
 & - \frac{q^{6g-6} t^{12g-12} (q^2 t - 1)^{2g} (qt - 1)^{2g}}{(q^2 t^2 - 1)(q^2 - 1)(qt^2 - 1)(q - 1)}.
 \end{aligned}$$

# Gothen's formula for $P(\mathcal{M}_{\text{Dol}}^{d,3}; t)$

$$\begin{aligned}
 P_t(\mathcal{M}) = & \frac{(1+t)^{4g-4}}{(1-t)^4} \left( 2t^2 + t^4 + 2t^{2g} + 2t^{2g+2} - \frac{1}{4}t^{4g-4} - 3gt^{4g-3} \right. \\
 & + (6g^2 + 2g - 3)t^{4g-2} + (11g - 12g^2)t^{4g-1} \\
 & \left. + (6g^2 - 10g + \frac{17}{4})t^{4g} - t^{8g-6} - t^{10g-8} \right) \\
 & + \frac{t^{2g}(1+t)^{2g-4}}{(1-t)^4(1+t^2)^2} \left( t^{6g-8}(1+t^3)^{2g}(-2g - t^2 + (2g-2)t^4) \right. \\
 & \left. + (1+t)^{2g}(-2t^4 - 2t^6 + t^{2g-4} + 2t^{2g-2} + t^{2g} - t^{4g-2}) \right) \\
 & - \frac{2^{2g}t^{2g}(1+t)^{2g-1}}{(1-t)^4} + \frac{2gt^{8g-8}(1+t)^{2g-3}(1+t^3)^{2g-1}}{(1-t)^3(1+t^2)} \\
 & + \frac{2^{2g-1}t^{10g-8}(1+t)^{2g}}{(1-t)^3(1-t^3)} + \frac{t^{4g-4}(1-t)^{2g-1}(1+t)^{2g-1}}{4(1+t^2)} \\
 & + \frac{t^{6g-2}(1+t)^{4g-3}(1+t^2+t^4)}{(t-1)^3(1+t^2)^2(t^6-1)} + \frac{(1+t^5)^{2g}(1+t^3)^{2g-1}}{(t^2-1)(t^4-1)^2(t^3-1)}
 \end{aligned}$$