Arithmetic and physics of Higgs moduli spaces

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Plan for lectures

Lecture 1: Cohomology of Higgs moduli via $\mathbb{C}^\times$ action

Lecture 2: Cohomology of character variety via arithmetic

Lecture 3: $P = W$ and refined Gopakumar-Vafa

Ubiquity of Higgs bundles

- [Hitchin 1987] introduces Higgs bundles (E, ϕ) on Riemann surfaces as solutions of a two-dimensional reduction of the 4D Yang–Mills equations $\rightsquigarrow$ Hitchin integrable system
- [Simpson 1990] studies Higgs bundles in the framework of non-Abelian Hodge theory of a variety
- [Beilinson–Drinfeld $\sim$1995] Geometric Langlands via quantization of Hitchin system
- [Donagi–Witten 1996] & [Moore–Nekrasov–Shatashvili 1997] importance of Higgs moduli in supersymmetric gauge theories
- [Kapustin–Witten 2005] S-duality perspective on Geometric Langlands and Higgs moduli spaces
- [Gaiotto–Neitzke–Moore 2008-] hyperkähler metric on Higgs moduli spaces maybe understood from wall-crossing
- [Ngô, 2010] symmetries of Hitchin fibers $\rightsquigarrow$ fundamental lemma
- [Chuang–Diaconescu–Pan 2010-] Donaldson–Thomas theory on local curve Calabi–Yau 3-folds $\rightsquigarrow$ Higgs moduli spaces

Moduli space of vector bundles

- C smooth complex projective curve of genus $g > 1$
- fix integers $r > 0$ and $d \in \mathbb{Z}$ always assume $(d, r) = 1$
- $\mathcal{N} :=$ moduli space of isomorphism classes of semi-stable rank r degree d vector bundles on C
- constructed using geometric invariant theory (GIT) or gauge theory
- vector bundle E is called *semi-stable* (*stable*) if every proper subbundle F satisfies

$$\mu(F) = \frac{\deg(F)}{\operatorname{rk}(F)} \overset{(<)}{\leq} \mu(E) = \frac{\deg(E)}{\operatorname{rk}(E)}$$

- when $(d, r) = 1$ semi-stability $\Leftrightarrow$ stability $\rightsquigarrow$ $\mathcal{N}^d$ is a non-singular projective variety
- $\mathbb{E}^d$ a universal vector bundle on $\mathcal{N}^d \times C$ in the sense that $\mathbb{E}^d|_{\{E\}\times C} \cong E$

Cohomology of $\mathcal{N}$

- $H^*(\mathcal{N}^d) := H^*(\mathcal{N}^d;\mathbb{C})$ well understood
- [Harder–Narasimhan 1975] and [Atiyah–Bott 1981] obtained recursive formulae for $P(\mathcal{N}^d;t) := \sum_{i=0} \dim(H^i(\mathcal{N}^d))t^i$
- $\rightsquigarrow P_t(\mathcal{N}^d)$ depends on d
- [Atiyah–Bott 1981] $\rightsquigarrow$ the ring $H^*(\mathcal{N}^d)$ is generated by the Künneth components of $c(\mathbb{E}) \in H^*(\mathcal{N}^d) \otimes H^*(C)$
- [Jeffrey–Kirwan 1998] $\rightsquigarrow$ all intersection numbers (integrals of products of universal generators)
  as conjectured by [Witten 1992]
- [Earl–Kirwan 2005] complete set of relations among the universal generators of $H^*(\mathcal{N}^d)$

Moduli space of Higgs bundles

- (E,ϕ) a Higgs bundle on C
  - E vector bundle on C
  - $\phi \in H^0(C; \mathrm{End}(E) \otimes K_C)$ *Higgs field*
- (E,ϕ) is called *semi-stable* (*stable*) if every proper ϕ-invariant subbundle $F \subset E$ satisfies

$$\mu(F) = \frac{\deg(F)}{\mathrm{rk}(F)} \overset{(<)}{\leq} \mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)}$$

- $\mathcal{M}^d :=$ moduli space of isomorphism classes of semi-stable rank r degree d Higgs bundles on C
- constructed using geometric invariant theory (GIT) or gauge theory
- $(d,r) = 1 \rightsquigarrow \mathcal{M}^d$ is non-singular, quasi-projective & $(\mathbb{E}, \mathbf{\Phi})$ universal Higgs bundle on $\mathcal{M}^d \times C$
- $\mathbb{C}^\times$ acts on $\mathcal{M}^d$ by $(E,\phi) \mapsto (E, \lambda\phi)$
  $\mathcal{M}^d$ is *semi-projective*

Semi-projective varieties

- $\mathcal{M}$ smooth quasi-projective variety *semi-projective*
  - has a $\mathbb{C}^\times$ action
  - $\mathcal{M}^{\mathbb{C}^\times}$ proper
  - $\lim_{\lambda\to 0} \lambda z$ exists for all $z \in \mathcal{M}$
- examples:
  - $\mathcal{M}^d$
  - semi-projective toric varieties
  - toric hyperkähler varieties
  - Nakajima quiver varieties
- $\mathcal{M}^{\mathbb{C}^\times} = \coprod_{i\in I} F_i$ connected components
- $U_i := \{z \in \mathcal{M} | \lim_{\lambda\to 0} \lambda z \in F_i\}$ affine bundle over F_i
- $\mathcal{M} = \coprod_{i\in I} U_i$ *Bialinicki-Birula decomposition*
- $D_i := \{z \in \mathcal{M} | \lim_{\lambda\to \infty} \lambda z \in F_i\}$ affine bundle over F_i
- $\mathcal{C} := \coprod_{i\in I} D_i \subset \mathcal{M}$ proper subvariety: *core* of $\mathcal{M}$

Cohomology of semi-projective varieties

- $H^*(\mathcal{M}) \cong \bigoplus_{i\in I} H^{*-2\mathrm{codim} F_i}(F_i)$ BB decomposition *perfect*
- $H^*_{\mathbb{C}^\times}(\mathcal{M}) \cong H^*(\mathcal{M}) \otimes H^*_{\mathbb{C}^\times}(pt)$ *equivariantly formal*
- $r : H^*_{\mathbb{C}^\times}(\mathcal{M}) \hookrightarrow H^*_{\mathbb{C}^\times}(\mathcal{M}^{\mathbb{C}^\times})$ *Kirwan injectivity*
- $\int_{\mathcal{M}} \alpha\beta := \sum_i \int_{F_i} \frac{i^*_{F_i}(\alpha\beta)}{E_{\mathbb{C}^\times}(N_{F_i})} \in H^*_{\mathbb{C}^\times}(\mathcal{M}) \otimes_{\mathbb{C}[u]} \mathbb{C}(u)$
  is a non-degenerate pairing on $H^*_{\mathbb{C}^\times}(\mathcal{M})$
- $Z := \mathcal{M}//_{\zeta_\infty}\mathbb{C}^\times$ orbifold
  $\overline{\mathcal{M}} := \mathcal{M} \times \mathbb{C}//_{\zeta_\infty}\mathbb{C}^\times = \mathcal{M} \coprod Z$ orbifold compactification
- $H^*(\overline{\mathcal{M}}) \twoheadrightarrow H^*(\mathcal{M})$ surjective
- $\mathcal{C} \subset \mathcal{M}$ is deformation retract
- $\rightsquigarrow H^*(\mathcal{M})$ has pure weight filtration

Bialiniczky-Birula decomposition for $\mathcal{M}^d$

- $[(E,\phi)] \in (\mathcal{M}^d)^{\mathbb{C}^\times}$ is fixed by $\mathbb{C}^\times$-action $\Leftrightarrow$ $(E,\phi) \cong (E,\lambda\phi)$
- $\rightsquigarrow E = E_1 \oplus \cdots \oplus E_k$

  and lower-triangular: $\phi|E_i \subset E_{i+1} \otimes K_C$ and $\phi|E_k = 0$
- its *type*: $(\mathrm{rk} E_1, \ldots, \mathrm{rk} E_k)$ ordered partition of r

  its *multi-degree* : $(\deg E_1, \ldots, \deg E_k)$ adding to d
- their locus is denoted $F^{d_1,\ldots,d_r}_{r_1,\ldots,r_k} \subset (\mathcal{M}^d)^{\mathbb{C}^\times}$

Examples for $F^{d_1,\dots,d_k}_{r_1,\dots,r_k}$

- fixed points of type $(r) \rightsquigarrow \phi = 0$ and E is stable $F^d_r \cong \mathcal{N}^d$
- fixed points of type $(1, 1, \dots, 1)$ are determined by E_1 and the divisors of zeros of $\phi_i \in H^0(E_i^{-1} \otimes E_{i+1} \otimes K_C)$

  $$\rightsquigarrow F^{d_1,\dots,d_r}_{1,\dots,1} \cong J^{d_1}(C) \times S^{d_2+2g-2-d_1}(C) \times \cdots \times S^{d_r+2g-2-d_{r-1}}(C)$$

- when $r = 2$, only two types:

  $F^d_2 \cong \mathcal{N}^d$ and

  $F^{d_1,d_2}_{1,1} \cong J^{d_1}(C) \times S^{2d_2+2g-2-d}(C)$ with stability $\rightsquigarrow d_2 < d/2$

Results on cohomology of $\mathcal{M}^d$

- Betti numbers ($b_i = \dim H^i(\mathcal{M}^d)$) computed for
  - $r = 2$ [Hitchin 1987] complicated formula (2 lines)
  - $r = 3$ [Gothen 1994] complicated formula (9 lines)
  - $r = 4$ [Garcia-Prada–Heinloth–Schmitt 2011] (7 lines & 20 summation signs)
  - $r > 4$ [Garcia-Prada–Heinloth 2012] claim computation converges
- universal generators for cohomology ring were proved by
  - $r = 2$ [Hausel–Thaddeus 2002]
  - $r > 2$ [Markman 2002]
- relations in cohomology ring described for $r = 2$ by [Hausel–Thaddeus 2002]
- equivariant intersection numbers conjectured by [Moore–Nekrasov–Shatashvili 1997] & [Hausel–Szenes 2008 $\leq$]