Arithmetic and physics in discrete algebraic geometry

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- many historical examples
- for example in the work of Minkowski (1864-1909)
- study of lattices in $\mathbb{R}^n \Leftrightarrow$ algebraic number theory in Minkowski's "Geometrie der Zahlen" (1896)
- (Minkowski 1907) introduces the Minkowski spacetime $\mathbb{R}^{3,1}$
- "The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality."

Frobenius on group characters

- FROBENIUS, F.G.: Über Gruppencharaktere (1896):

  " I shall develop the concept [of character for arbitrary finite groups] here in the belief that through its introduction, group theory will be substantially enriched. "

- After proving the orthogonality relations ($k = 2; g = 0$ below) Frobenius' first application was the $g = 0$ case of

Theorem (Frobenius 1896, Hurwitz 1902, Freed-Quinn 1993,...)

Let $\mathcal{C}_1, \ldots, \mathcal{C}_k \subset \mathrm{G}$ be conjugacy classes in a finite group G then

$$\#\{c_i \in \mathcal{C}_i, a_j, b_j \in \mathrm{G} | c_1 c_2 \cdots c_k [a_1, b_1] \ldots [a_g, b_g] = 1\} =$$

$$= \sum_{\chi \in \mathrm{Irr}(\mathrm{G})} \frac{|\mathrm{G}|^{2g-1}}{\chi(1)^{2g-2}} \prod_{i=1}^{k} \frac{\chi(\mathcal{C}_i)|\mathcal{C}_i|}{\chi(1)}$$

Arithmetic harmonic analysis for $\mathrm{GL}_n(\mathbb{F}_q)$

- $\mathrm{G} = \mathrm{GL}_n(\mathbb{F}_q)$
- character table of $\mathrm{GL}_n(\mathbb{F}_q)$ was calculated by
  (Jordan, Schur, 1907) for $n = 2$
  (Steinberg, 1951) for $n = 3, 4$
  (Green, 1955) for all n
- (Hausel–Letellier–Villegas, 2008) calculated explicitly (using Macdonald polynomials)

$$\sum_{\chi \in \mathrm{Irr}(\mathrm{GL}_n(\mathbb{F}_q))} \frac{|\mathrm{G}|^{2g-1}}{\chi(1)^{2g-2}} \prod_{i=1}^{k} \frac{\chi(\mathcal{C}_i)|\mathcal{C}_i|}{\chi(1)} =$$

$$\#\{\mathrm{c}_i \in \mathcal{C}_i, a_j, b_j \in \mathrm{G} | c_1 c_2 \cdots c_k [a_1, b_1] \ldots [a_g, b_g] = 1\},$$

where $\mathcal{C}_i \subset \mathrm{GL}_n(\mathbb{F}_q)$ are generic semisimple conjugacy classes

Character table of $\mathrm{GL}_2(\mathbb{F}_q)$

Table 1: characters of $\mathbf{GL}_2(\mathbb{F}_q)$
(note that $|\mathbf{GL}_2(\mathbb{F}_q)| = q(q-1)^2(q+1)$)

Classes	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ $a \in \mathbb{F}_q^\times$	$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ $a, b \in \mathbb{F}_q^\times$ $a \neq b$	$\begin{pmatrix} x & 0 \\ 0 & {}^F x \end{pmatrix}$ $x \in \mathbb{F}_{q^2}^\times$ $x \neq {}^F x$	$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ $a \in \mathbb{F}_q^\times$
Number of classes of this type	$q-1$	$\frac{(q-1)(q-2)}{2}$	$\frac{q(q-1)}{2}$	$q-1$
Cardinal of the class	1	$q(q+1)$	$q(q-1)$	q^2-1
$R_{\mathbf{T}}^{\mathbf{G}}(\alpha,\beta)$ $\alpha,\beta \in \mathrm{Irr}(\mathbb{F}_q^\times)$ $\alpha \neq \beta$	$(q+1)\alpha(a)\beta(a)$	$\alpha(a)\beta(b)+$ $\alpha(b)\beta(a)$	0	$\alpha(a)\beta(a)$
$-R_{\mathbf{T}_s}^{\mathbf{G}}(\omega)$ $\omega \in \mathrm{Irr}(\mathbb{F}_{q^2}^\times)$ $\omega \neq \omega^q$	$(q-1)\omega(a)$	0	$-\omega(x)-\omega({}^F x)$	$-\omega(a)$
$\mathrm{Id}_{\mathbf{G}}.(\alpha \circ \det)$ $\alpha \in \mathrm{Irr}(\mathbb{F}_q^\times)$	$\alpha(a^2)$	$\alpha(ab)$	$\alpha(x.{}^F x)$	$\alpha(a^2)$
$\mathrm{St}_{\mathbf{G}}.(\alpha \circ \det)$ $\alpha \in \mathrm{Irr}(\mathbb{F}_q^\times)$	$q\alpha(a^2)$	$\alpha(ab)$	$-\alpha(x.{}^F x)$	0

Example $\mathrm{GL}_2(\mathbb{F}_q)$

- $\mathrm{G} = \mathrm{GL}_2(\mathbb{F}_q)$, $k = 1$ $\mathcal{C}_1 = \{-1\} \subset \mathrm{GL}_2(\mathbb{F}_q)$
- $$\frac{1}{|\mathrm{PGL}_2(\mathbb{F}_q)|(q-1)^{2g}} \#\{a_j, b_j \in \mathrm{G} | [a_1, b_1] \ldots [a_g, b_g] = -1\} =$$

$$\frac{1}{|\mathrm{PGL}_2(\mathbb{F}_q)|(q-1)^{2g}} \sum_{\chi \in \mathrm{Irr}(\mathrm{GL}_2(\mathbb{F}_q))} \frac{|\mathrm{G}|^{2g-1}}{\chi(1)^{2g-2}} \frac{\chi(-1)}{\chi(1)} =$$

$$(q^2-1)^{2g-2} + q^{2g-2}(q^2-1)^{2g-2} - \frac{1}{2} q^{2g-2}(q-1)^{2g-2} - \frac{1}{2} q^{2g-2}(q+1)^{2g-2}.$$

- e.g. $g = 0$ gives 0 when $g = 1$ it gives 1

Example $\mathrm{SL}_n(\mathbb{F}_q)$

- character table for $\mathrm{SL}_2(\mathbb{F}_q)$ by (Jordan 1907), (Schur 1907) ... for $\mathrm{SL}_n(\mathbb{F}_q)$ (Bonnafé 2006), (Shoji 2006)
- for $\mathrm{G} = \mathrm{SL}_2(\mathbb{F}_q)$, $k = 1$, $\mathcal{C}_1 = \{-1\} \subset \mathrm{SL}_2(\mathbb{F}_q)$
-

$$\frac{1}{|\mathrm{PGL}_2(\mathbb{F}_q)|}\#\{a_j, b_j \in \mathrm{G} | [a_1, b_1] \dots [a_g, b_g] = -1\} =$$

$$\frac{1}{|\mathrm{PGL}_2(\mathbb{F}_q)|} \sum_{\chi \in \mathrm{Irr}(\mathrm{SL}_2(\mathbb{F}_q))} \frac{|\mathrm{G}|^{2g-1}}{\chi(1)^{2g-2}} \frac{\chi(-1)}{\chi(1)} =$$

$$(q^2-1)^{2g-2}+q^{2g-2}(q^2-1)^{2g-2}-\frac{1}{2}q^{2g-2}(q-1)^{2g-2}-\frac{1}{2}q^{2g-2}(q+1)^{2g-2}+$$
$$+(2^{2g}-1)q^{2g-2}\left(\frac{(q-1)^{2g-2}-(q+1)^{2g-2}}{2}\right).$$

Character table of $\mathrm{SL}_2(\mathbb{F}_q)$

Table 2: characters of $\mathbf{SL}_2(\mathbb{F}_q)$ for q odd
(note that $|\mathbf{SL}_2(\mathbb{F}_q)| = q(q-1)(q+1)$)

Classes	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ $a \in \{1,-1\}$	$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ $a \in \mathbb{F}_q^\times$ $a \neq \{1,-1\}$	$\begin{pmatrix} x & 0 \\ 0 & {}^F x \end{pmatrix}$ $x.{}^F x = 1$ $x \neq {}^F x$	$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ $a \in \{1,-1\}$, $b \in \{1, x\}$ with $x \in \mathbb{F}_q^\times - (\mathbb{F}_q^\times)^2$
Number of classes of this type	2	$(q-3)/2$	$(q-1)/2$	4
Cardinal of the class	1	$q(q+1)$	$q(q-1)$	$(q^2-1)/2$
$R_{\mathbf{T}}^{\mathbf{G}}(\alpha)$ $\alpha \in \mathrm{Irr}(\mathbb{F}_q^\times)$ $\alpha^2 \neq \mathrm{Id}$	$(q+1)\alpha(a)$	$\alpha(a) + \alpha(\frac{1}{a})$	0	$\alpha(a)$
$\chi_{\alpha_0}^{\varepsilon}$ $\varepsilon \in \{1,-1\}$	$\frac{q+1}{2}\alpha_0(a)$	$\alpha_0(a)$	0	$\frac{\alpha_0(a)}{2}(1- \varepsilon\alpha_0(ab)\sqrt{\alpha_0(-1)q})$
$-R_{\mathbf{T}_s}^{\mathbf{G}}(\omega)$ $\omega \in \mathrm{Irr}(\mu_{q+1})$ $\omega^2 \neq \mathrm{Id}$	$(q-1)\omega(a)$	0	$-\omega(x) - \omega({}^F x)$	$-\omega(a)$
$\chi_{\omega_0}^{\varepsilon}$ $\varepsilon \in \{1,-1\}$	$\frac{q-1}{2}\omega_0(a)$	0	$-\omega_0(x)$	$\frac{\omega_0(a)}{2}(-1+ \varepsilon\alpha_0(ab)\sqrt{\alpha_0(-1)q})$
$\mathrm{Id}_{\mathbf{G}}$	1	1	1	1
$\mathrm{St}_{\mathbf{G}}$	q	1	-1	0

Character varieties for GL_n and SL_n

- fix integers $n > 1$ and d such that $(n, d) = 1$ and ζ_n primitive nth root of unity; coefficients in $\mathbb{C}$ or $\overline{\mathbb{F}}_q$ or $\mathbb{Z}[\zeta_n]$
- the GL_n-character variety:

$$\mathcal{M} := \{(A_i, B_i)_{i=1..g} \in \mathrm{GL}_n^{2g} \mid [A_1, B_1] \dots [A_g, B_g] = \zeta_n^d I_n\} // \mathrm{PGL}_n$$

non-singular, affine

- the SL_n-character variety:

$$\check{\mathcal{M}} := \{(A_i, B_i)_{i=1..g} \in \mathrm{SL}_n^{2g} \mid [A_1, B_1] \dots [A_g, B_g] = \zeta_n^d I_n\} // \mathrm{PGL}_n$$

non-singular, affine

Character variety for PGL_n

- $(\mathrm{GL}_1)^{2g}$ acts on $\mathcal{M}$
- $\Gamma \cong (\mathbb{Z}_n)^{2g} \subset (\mathrm{GL}_1)^{2g}$ acts on $\check{\mathcal{M}}$
- the PGL_n-character variety: $\hat{\mathcal{M}} := \check{\mathcal{M}}/\Gamma \cong \mathcal{M}/(\mathrm{GL}_1)^{2g}$ is an affine orbifold

Character variety over a finite field

- Frobenius' character formula
  - for $\mathrm{GL}_n(\mathbb{F}_q) \rightsquigarrow \#\mathcal{M}(\mathbb{F}_q) = \sum_{\chi \in \mathrm{Irr}(\mathrm{GL}_n(\mathbb{F}_q))} \frac{|\mathrm{GL}_n(\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-2}} \frac{\chi(\xi_n^d I_n)}{\chi(1)}$
  - for $\mathrm{SL}_n(\mathbb{F}_q) \rightsquigarrow \#\check{\mathcal{M}}(\mathbb{F}_q) = \sum_{\chi \in \mathrm{Irr}(\mathrm{SL}_n(\mathbb{F}_q))} \frac{|\mathrm{SL}_n(\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-2}} \frac{\chi(\xi_n^d I_n)}{\chi(1)}$
  - for $\mathrm{PGL}_n(\mathbb{F}_q) \rightsquigarrow \#\hat{\mathcal{M}}(\mathbb{F}_q) = \frac{\#\mathcal{M}(\mathbb{F}_q)}{(q-1)^{2g}}$
- in all these cases the count is a polynomial in q
- (Katz 2008) $\rightsquigarrow$ if for a variety $\#(X(\mathbb{F}_q)) \in \mathbb{Z}[q]$ is a polynomial, then

$$\#(X(\mathbb{F}_q)) = E(X;q) = \sum \dim(W_i/W_{i-1}(H_c^k(X/\mathbb{C};\mathbb{Q})))(-1)^k q^{\frac{i}{2}}$$

is the *Serre polynomial*

Topological Mirror Test

- X non-singular algebraic variety/$\mathbb{C}$, Γ finite group acting on X
- define stringy Serre-polynomial of the orbifold X/Γ by

$$E^B_{st}(X/\Gamma;q) = \sum_{[\gamma]\in[\Gamma]} E(X^\gamma/C(\gamma), L^B_\gamma; q)(q)^{F(\gamma)}$$

- motivating: (Kontsevich 1995) for $Y \to X/\Gamma$ crepant $\rightsquigarrow$ $E_{st}(X/\Gamma;q) = E(Y;q)$
- recall that the PGL_n-character variety $\hat{\mathcal{M}} = \check{\mathcal{M}}/\Gamma$ is an orbifold with $\Gamma \cong (\mathbb{Z}_n)^{2g}$
- (Hausel–Thaddeus 2001, Hausel–Villegas 2004) character varieties for Langlands dual groups are "mirror symmetric"

Conjecture (Hausel–Villegas 2004, Topological Mirror Test)

$$E(\check{\mathcal{M}};q) = E^B_{\mathrm{st}}(\hat{\mathcal{M}};q).$$

Example

- when $n = 2$

$$E(\check{\mathcal{M}}; q) - E(\hat{\mathcal{M}}; q) = (2^{2g} - 1)q^{2g-2}\left(\frac{(q-1)^{2g-2} - (q+1)^{2g-2}}{2}\right)$$

- $\check{\mathcal{M}}^{\gamma}$ can be identified with $(\mathbb{C}^{\times})^{2g-2} \rightsquigarrow$

$$E(\check{\mathcal{M}}^{\gamma}/\Gamma, L_{B,\gamma}; q) = \frac{(q-1)^{2g-2} - (q+1)^{2g-2}}{2}$$

- implies Topological Mirror Test when $n = 2$
- similar argument settles $n = p$
- ongoing work (Hausel-Mereb-Villegas 2012) handles all n $\rightsquigarrow$ character formulae reminiscent of the fundamental lemma for SL_n in the Langlands program

Mirror symmetry for Langlands dual Hitchin systems

- by the non-Abelian Hodge theorem $\mathcal{M} \overset{diff}{\simeq} \mathcal{M}_{\mathrm{Dol}}$ with moduli Higgs bundles $\rightsquigarrow$ Hitchin map $\chi : \mathcal{M}_{\mathrm{Dol}} \to \mathcal{A}$
- $$\begin{array}{ccc} \check{\mathcal{M}} & & \hat{\mathcal{M}} \\ & {\scriptstyle \check{\chi}} \searrow \quad \swarrow {\scriptstyle \hat{\chi}} & \\ & \mathcal{A} & \end{array}$$

  $\rightsquigarrow$ SYZ construction for mirror symmetric Calabi-Yau's
  $\rightsquigarrow$ $\check{\mathcal{M}}$ and $\hat{\mathcal{M}}$ could be considered mirror symmetric!
  $\rightsquigarrow$ can be deduced from (Kapustin-Witten 2006) S-duality
- Topological Mirror Test is the agreement of Hodge numbers
  $\rightsquigarrow$ relative of Ngô's geometric fundamental lemma
- for $n = 2$ we proved Topological Mirror Test from certain patterns in $Irr(\mathrm{SL}_2(\mathbb{F}_q))$ vs. $Irr(\mathrm{GL}_2(\mathbb{F}_q))$
  due to (Schur 1907) and (Jordan 1907)

Jordan's character table of $\mathrm{PGL}_2(\mathbb{F}_q)$

The Binary Linear Fractional Group F_1 *in the* $GF[p^n]$, $p > 2$, *of all Determinants not Zero.*

Below is given the table of group-characters.

N	1	1	1	1	$\frac{s-3}{2}$	$\frac{s-1}{2}$
χ_λ	1	1	s	s	$s+1$	$s-1$
χ_μ	1	1	0	0	1	-1
$\chi(R^{2a})$	1	1	1	1	$r^{2a}+r^{-2a}$	0
$\chi(S^{2b})$	1	1	-1	-1	0	$-t^{2b}-t^{-2b}$
$\chi(R^{2a+1})$	1	-1	1	-1	$r^{2a+1}+r^{-(2a+1)}$	0
$\chi(S^{2b+1})$	1	-1	-1	1	0	$-t^{2b+1}-t^{-(2b+1)}$

where r and t are the roots (except ± 1) of $r^{s-1}=1$ and $t^{s+1}=1$ respectively. As before $e=f$.

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Schur's character table of $\mathrm{SL}_2(\mathbb{F}_q)$

2. Die Gruppe $\mathfrak{L}_{p^n}$, die durch die ganzen linearen Substitutionen

$$\xi_1 = \alpha\eta_1 + \beta\eta_2, \quad \xi_2 = \gamma\eta_1 + \delta\eta_2$$

gebildet wird, deren Determinante gleich 1 ist. — Die Ordnung der Gruppe

Die $s+4$ Charaktere von $\mathfrak{L}_s$ lassen sich in folgender Tabelle zusammenfassen:

	1	1	$\frac{s-3}{2}$	$\frac{s-1}{2}$	2	2
$\chi(E)$	1	s	$s+1$	$s-1$	$\frac{1}{2}(s+1)$	$\frac{1}{2}(s-1)$
$\chi(F)$	1	s	$(-1)^{\alpha}(s+1)$	$(-1)^{\beta}(s-1)$	$\frac{\varepsilon}{2}(s+1)$	$-\frac{\varepsilon}{2}(s-1)$
$\chi(P)$	1	0	1	-1	$\frac{1}{2}(1 \pm \sqrt{\varepsilon s})$	$\frac{1}{2}(-1 \pm \sqrt{\varepsilon s})$
$\chi(Q)$	1	0	1	-1	$\frac{1}{2}(1 \mp \sqrt{\varepsilon s})$	$\frac{1}{2}(-1 \mp \sqrt{\varepsilon s})$
$\chi(A^a)$	1	1	$\varrho^{\alpha a} + \varrho^{-\alpha a}$	0	$(-1)^a$	0
$\chi(B^b)$	1	-1	0	$-(\sigma^{\beta b} + \sigma^{-\beta b})$	0	$-(-1)^b$.

*) Vgl. *Dickson*, Linear Groups, Cap. XII.